Answer all questions.

1. For any group $G$, define $G_1 = G$ and $G_{n+1} = (G, G_n)$. Call $G$ nilpotent if $G_N = \{1\}$ for some $N$. Prove that any group of prime power order is nilpotent.

2. Let $K$ be a field and $G$ a finite group.
   
   a) Prove that the number of one-dimensional $K$-representations of $G$ is (up to equivalence) at most $[G : G']$.
   
   b) Show with an example that the inequality can be strict.

3. Find all prime ideals of $\mathbb{F}_3[x, y]/(y^2 - x^3 + x)$ whose intersection with $\mathbb{F}_3[x]$ is equal to $(x^2 + x + 2)$.

4. a) Suppose $R$ is a ring and $\phi : A \to B$ is a surjective homomorphism of left $R$-modules. Prove that for any right $R$-module $M$, $\phi \otimes \text{id} : A \otimes_R M \to B \otimes_R M$ is also surjective.

   b) Give an example to show that the above is false if both occurrences of "surjective" are replaced with "injective".

5. Suppose $K/F$ is a normal algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

6. Find an explicit decomposition into direct product of matrix algebras over division rings of $M_2(\mathbb{F}_4) \otimes_{\mathbb{F}_2} M_2(\mathbb{F}_4)$. 