Algebra Preliminary Exam, June 2011

1. (a) Show that any group of order 42 contains a normal subgroup of index 6.
   (b) Find (with proof) a group of order 42 that contains a subgroup $H$ such that $H \cong S_3$ and such that $H$ is not normal.

2. Let $k$ be a field.
   (a) Prove that any semi-simple $k$-algebra of dimension $\leq 3$ is commutative.
   (b) Does the result of (ii) remain true if you omit the hypothesis of semi-simplicity?

3. Let $G$ be a finite group and let $\rho$ be an representation of $G$ on an $n$-dimensional vector space $V$ over the field $\mathbb{C}$ of complex numbers. Suppose that for every element $g \in G$, there exists a basis for $V$ with respect to which the linear automorphism $\rho(g)$ has the form

   \[
   \begin{pmatrix}
   1 & * & \cdots & * \\
   0 & 1 & \cdots & * \\
   0 & 0 & 1 & \cdots & * \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & \cdots & 1
   \end{pmatrix}
   \]

   (i.e. 0s below the diagonal, 1s on the diagonal, and unspecified entries above the diagonal). Prove that $\rho$ is trivial, i.e. that $\rho(g)$ acts as the identity on $V$ for every $g \in G$.

4. Find the Galois group of $x^4 + 1$ over each of the following fields: $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{F}_3$, $\mathbb{F}_5$.

5. Let $A$ be a commutative ring with 1, and let $M$ be a finitely generated $A$-module.
   (a) If $m$ is a maximal ideal of $A$, prove that $M/mM$ is non-zero if and only if the localization $M_m$ is non-zero.
   (b) Is the analogous statement true if we replace $m$ by a non-maximal prime ideal of $A$? Carefully explain why or why not.

6. Suppose that $A$ is a commutative ring with 1.
   (a) If $N \subset M$ are $A$-modules and $N_m = M_m$ for all maximal ideals $m$, show that $N = M$.
   (b) Suppose that $A$ has only finitely many maximal ideals. If $A_m$ is Noetherian for all maximal ideals $m$, show that $A$ is Noetherian.
7. Let \( A = \mathbb{C}[x, y, z]/(x^2 + y^2 - 2z^2) \), and let \( I \) be the ideal of \( A \) generated by \( x - y \).

(a) Prove that \( I \) is a radical ideal.
(b) Find all the minimal primes of \( I \).
(c) Determine the height of each of the prime ideals you found in (b).