Algebra Preliminary Exam, September 14, 2001

Give sufficient detail to justify your answers.

1. Let F denote the field $\mathbb{F}_{25}(x)$, the field extension of pure transcendence degree 1 over the finite field with 25 elements. Exhibit a Galois extension of F of degree 25 and determine its Galois group. Exhibit a field extension of F which is inseparable.

2. Let A_6 denote the alternating group of 6 letters.
   a.) Exhibit a Sylow 3-subgroup of A_6.
   b.) What is the structure of this Sylow subgroup?
   c.) How many Sylow 3-subgroups does A_6 have?

3. Let F be a field and let R denote the ring of 5×5 matrices with coefficients in F.
   a.) Show that R is Artinian.
   b.) Determine the Jacobson radical of R.
   c.) Determine all irreducible R-modules.

4. Give examples of the following:
   a.) A non-trivial ring for which all modules are flat.
   b.) A ring R and an R-module M which is not flat.
   c.) A ring R and a flat R-module which is not free.

5. Let A be an integral domain, $S \subset A - \{0\}$.
   a.) Construct $S^{-1}A$.
   b.) Prove a universal property for $A \to S^{-1}A$ which characterizes $S^{-1}A$ up to canonical isomorphism.
   c.) Give a necessary and sufficient condition for an A-module M to have its A-module structure extend to an $S^{-1}A$-module structure.

6. Let $\mathbb{C}$ be the complex number field and let $R = \mathbb{C}[x, y]$ be the polynomial ring on two generators over $\mathbb{C}$.
   a.) Describe the maximal ideals of R.
   b.) Describe all the prime ideals of R.

7. Consider the cyclotomic field $F = \mathbb{Q}[2^{1/3}, \zeta_3]$ where $\zeta_3 = e^{2\pi i/3}$. Then $F/\mathbb{Q}$ is Galois with Galois group the symmetric group Σ_3. Find all the subfields of K. Which of these are normal extensions of $\mathbb{Q}$? Which of these are separable extensions of $\mathbb{Q}$?

8. Let $R = F[x]$ be a polynomial ring on one generator over a field, and view F as an R-module via the ring quotient map $R \to F, x \mapsto 0$.
   a.) For any R-module M, describe $M \otimes_R F$.
   b.) Give an R-module M such that $Tor_R(M, F) \neq 0$.
   c.) Give an R-module $M \neq 0$ such that $Tor_r(M, F) = 0$.