ALGEBRA PRELIM – September 13, 2006

(1) Let $\mathbb{F}$ be a field of cardinality 625.
- What is $\mathbb{F}$ as an additive group?
- What is the multiplicative group $\mathbb{F}^*$ of units of $\mathbb{F}$?
- What is the group of field automorphisms of $\mathbb{F}$?

(2) Let G be the cyclic group $\mathbb{Z}/5$ and let $\mathbb{F}$ be the field of 625 elements.
- Describe the group algebra $\mathbb{F}G$ as a k-vector space and specify its ring structure.
- Let $\mathbb{C}$ denote the complex numbers and let $\mathbb{C}G$ be the group algebra of G over $\mathbb{C}$. Decompose $\mathbb{C}G$ as a direct sum of irreducible $\mathbb{C}G$ modules.
- List the simple $\mathbb{F}G$-modules and determine the Jacobson radical of $\mathbb{F}G$.

(3) Consider the ring $R = \mathbb{C}[x,y]/(y^2 - x^3)$, where $\mathbb{C}$ denotes the complex numbers.
- Show that R is an integral domain.
- Find a chain of prime ideals of R of maximal length.
- Describe all the maximal ideals of R.
- Construct the integral closure of R.

(4) Let G be a group of order 63.
- Show that every 7-Sylow subgroup G_7 of G is normal
- Show that G must be a semi-direct product.
- List all isomorphism classes of groups of order 63.

(5) Let R be a ring, M a right R-module, and N a left R-module.
- State the universal mapping property of $M \otimes_R N$.
- Give an example of a commutative ring R and non-zero R-modules M, N with $M \otimes_R N = 0$.
- Let K be a field and let A, B be K-algebras. Describe the natural K-algebra structure on $A \otimes_K B$.

(6) Let $K = \mathbb{Q}[i]$.
- Show that K is a field.
- Determine the degree of the splitting field L of $x^{15} - 1$ over K.
- Determine $Gal(L/K)$.
- Describe the fields intermediate between K and L.