ALGEBRA PRELIM – September 13, 2006

(1) Let \( F \) be a field of cardinality 625.
   - What is \( F \) as an additive group?
   - What is the multiplicative group \( F^* \) of units of \( F \)?
   - What is the group of field automorphisms of \( F \)?

(2) Let \( G \) be the cyclic group \( \mathbb{Z}/5 \) and let \( F \) be the field of 625 elements.
   - Describe the group algebra \( FG \) as a \( k \)-vector space and specify its ring structure.
   - Let \( \mathbb{C} \) denote the complex numbers and let \( \mathbb{C}G \) be the group algebra of \( G \) over \( \mathbb{C} \). Decompose \( \mathbb{C}G \) as a direct sum of irreducible \( \mathbb{C}G \) modules.
   - List the simple \( FG \)-modules and determine the Jacobson radical of \( FG \).

(3) Consider the ring \( R = \mathbb{C}[x,y]/(y^2 - x^3) \), where \( \mathbb{C} \) denotes the complex numbers.
   - Show that \( R \) is an integral domain.
   - Find a chain of prime ideals of \( R \) of maximal length.
   - Describe all the maximal ideals of \( R \).
   - Construct the integral closure of \( R \).

(4) Let \( G \) be a group of order 63.
   - Show that every 7-Sylow subgroup \( G_7 \) of \( G \) is normal.
   - Show that \( G \) must be a semi-direct product.
   - List all isomorphism classes of groups of order 63.

(5) Let \( R \) be a ring, \( M \) a right \( R \)-module, and \( N \) a left \( R \)-module.
   - State the universal mapping property of \( M \otimes_R N \).
   - Give an example of a commutative ring \( R \) and non-zero \( R \)-modules \( M, N \) with \( M \otimes_R N \neq 0 \).
   - Let \( K \) be a field and let \( A, B \) be \( K \)-algebras. Describe the natural \( K \)-algebra structure on \( A \otimes_K B \).

(6) Let \( K = \mathbb{Q}[i] \).
   - Show that \( K \) is a field.
   - Determine the degree of the splitting field \( L \) of \( x^{15} - 1 \) over \( K \).
   - Determine \( \text{Gal}(L/K) \).
   - Describe the fields intermediate between \( K \) and \( L \).