ALGEBRA PRELIM – SEPTEMBER 21, 2007

1. (i) Compute the Galois group of the polynomial $X^6 + 3$ over $\mathbb{Q}$, and describe how its elements act on the roots of this polynomial.

(ii) Find a primitive element for the splitting field of $X^6 + 3$.

(iii) List all fields intermediate between $\mathbb{Q}$ and the splitting field of $X^6 + 3$.

2. (i) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Prove that if A and C are torsion, then the same is true of B.

(ii) Let $0 \to A \to B \to C \to D \to 0$ be an exact sequence of abelian groups. If A and D are torsion, is it necessarily true that B and C are also torsion? If so, prove it. If not, give a counter-example.

3. Let A be a commutative local ring, with unique maximal ideal $\mathfrak{m}$, and residue field $k := A/\mathfrak{m}$. Let M be a faithful, finitely generated A-module.

(i) If $M/\mathfrak{m}M$ is 1-dimensional over k, prove that M is free of rank 1 over A.

(ii) If $M/\mathfrak{m}M$ is 2-dimensional over k, is M necessarily free over A? If so, prove it. If not, give a counter-example.

(iii) Does the analogue of (i) hold if omit either of the hypotheses that M is faithful or finitely generated? For each hypothesis, either prove that it may be omitted, or else provide a counter-example showing that it is necessary.

5. Let I denote the ideal (XY, XZ, YZ) of $\mathbb{C}[X, Y, Z]$.

(i) What are the minimal prime ideals of I?

(ii) Is the quotient ring $\mathbb{C}[X, Y, Z]/I$ reduced?

5. Let G_1 and G_2 be two groups, both of order 128, let X_1 and X_2 be two sets, both of order 8, and suppose given a faithful action of G_1 on X_1 and a faithful action of G_2 on X_2. Prove that there exists an isomorphism $\phi : G_1 \xrightarrow{\sim} G_2$, and a bijection $\psi : X_1 \xrightarrow{\sim} X_2$, such that $\psi(g \cdot x) = \phi(g) \cdot \psi(x)$ for every $g \in G_1$ and $x \in X_1$.

6. List all semi-simple $\mathbb{R}$-algebras of dimension 4 whose centre is:

(i) 1-dimensional

(ii) 2-dimensional

(iii) 3-dimensional.

(iv) 4-dimensional.

(v) Which (if any) of the semi-simple $\mathbb{R}$-algebras that you have found is isomorphic to a group algebra $\mathbb{R}[G]$ for some group G?