Preliminary Examination — Algebra — Sept. 10, 1999

Do as many problems or parts of problems as you can. You need not do every problem to pass.

Since time is limited, you will be allowed some freedom to skip details, provided you include all *important relevant points.* In general you are allowed to use any standard results you could have been expected to learn in a first year graduate algebra course *as long as what you use is not tantamount to assuming what the problem asks you to prove.*

1. Let E denote the group of all functions $f : \mathbf{R}^2 \to \mathbf{R}^2$ of the form $f(x) = Ax + b$ where A is a 2×2 rotation matrix (i.e., an orthogonal matrix of determinant one), and $b \in \mathbf{R}^2$. Let S be the subgroup of those f with $b = 0$ (i.e., rotations about the origin), and let T be the subgroup of those f with $A = I$. You may assume E is a group under composition of functions and S and T are subgroups.
(a) Show that T is a normal subgroup.
(b) Identify geometrically gSg^{-1} where g is an element of T.
(c) Show that E is the semi-direct product of S with T.

2. Let $p < q$ be primes. Show that every group of order pq is solvable.

3. Let A be the abelian group (using additive notation for the group operation) generated by $\{x_1, x_2, x_3\}$ and subject to the relations

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 + 2x_2 - x_3 &= 0 \\ x_1 + 3x_2 + 3x_3 &= 0 \end{aligned}$$

Prove or disprove the statement: A is cyclic.

4. Let $f(X) = X^8 - 1$. Find the Galois group of $f(X)$ over the following base fields.
(a) The rational field $\mathbf{Q}$.
(b) The field $\mathbf{Q}[\sqrt{2}]$.
(c) The field $\mathbf{F}_3$ of three elements.

5. Let A be a commutative ring. For each of the following statements, explain *briefly* why it is true or give a counterexample with an explanation of why it is a counterexample.
(a) If A is a unique factorization domain, then A is a principal ideal domain.
(b) If A is a principal ideal domain, then A is a unique factorization domain.
(c) If A is a Euclidean domain, then A is a unique factorization domain.

6. Find all semisimple rings with 36 elements. Explain which major theorems you are using.

7. Let A be a commutative local ring with maximal ideal M. Suppose L is a finitely generated A-module and $f : L \to L$ an A-endomorphism.
(a) Show that f induces an endomorphism $\overline{f} : L/ML \to L/ML$.
(b) Show that if $\overline{f}$ is an epimorphism, then f is an epimorphism.

8. Let V be a finite dimensional vector space over a field k, and let $V^* = \mathrm{Hom}_k(V, k)$ be its dual. Let $A = \mathrm{Hom}_k(V, V)$ be the endomorphism ring of V. Let A act on V by $fx = f(x)$ for $f \in A$ and x in V, and let A act on V^* by $(\alpha f)(x) = \alpha(f(x))$ for $x \in V, \alpha \in V^*$, and $f \in A$. You may assume these actions make V a left A-module and V^* a right A-module.
(a) Consider the pairing $(\alpha, x) \mapsto \alpha(x)$ where $x \in V$, and $\alpha \in V^*$. Show that this defines a homomorphism $V^* \otimes_A V \to k$.
(b) Show that the homomorphism defined in (a) is an isomorphism.