PRELIMINARY EXAM IN ANALYSIS FALL 2015

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Let E be a Lebesgue measurable subset of $\mathbb{R}^d$.
  (a) Define what it means for a function $f : E \to \mathbb{R}$ to be measurable.
  (b) Show that if f is measurable then so is $|f|$.
  (c) Let f and g be measurable functions defined on E. Show that $f + g$ is measurable.

(2) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space.
  (a) State the Monotone Convergence Theorem.
  (b) Show that if $f_1, f_2, \ldots$ are nonnegative measurable functions on X then

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

  (c) Suppose $E_1, E_2, \ldots$ are measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Using part (b), show that for almost every $x \in X$, the set $\{n \in \mathbb{N} \mid x \in E_n\}$ is finite.

(3) Let f be an integrable function on $\mathbb{R}^d$ with respect to the Lebesgue measure m. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| dm < \varepsilon$ whenever E is a measurable set with $m(E) < \delta$.

(4) Let $\{f_n\}_{n \geq 1}$ be a sequence of functions in $L^1(\mathbb{R})$ with respect to the Lebesgue measure m. Suppose that f_n converges **pointwise** to a function $f \in L^1(\mathbb{R})$. Under each of the following assumptions, does f_n converge to f in the L^1 norm?
  (a) $|f_n| \leq 1$.
  (b) $\mathrm{supp}(f_n)$, the support of f_n, is contained in $[0, 1]$.
  (c) Both (a) and (b) hold.
  (d) $m(\mathrm{supp}(f_n)) \leq 1$ and $|f_n| \leq 1$.
In each case, you must give a proof or a counterexample. You may quote theorems without proof.

(5) Let f be a measurable function on a σ-finite measure space $(X, \mathcal{M}, \mu)$, with $f > 0$ almost everywhere. Show that if E is a measurable set with $\int_E f d\mu = 0$, then $\mu(E) = 0$.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$ denote the Fourier transform on $\mathbb{R}$.

  (a) Show that there exists a unique function $g \in L^2(\mathbb{R}, dx)$ such that $\mathcal{F}g(x) = e^{-|x|}$.
  (b) Calculate $||g||_{L^2}$.
  (c) Is $g \in C^{\infty}(\mathbb{R}) \cap L^2(\mathbb{R})$? Prove that your answer is correct.
  (d) Is g of rapid decay, i.e., is it true that for all nonnegative integer m there is a constant C_m such that $|g(y)| \leq C_m(1 + |y|)^{-m}$ for all $y \in \mathbb{R}$?

(2) Let H be a separable Hilbert space. Prove from scratch (without quoting theorems from a text) that every bounded linear functional $\Lambda : H \to \mathbb{C}$ is given by the inner product with some vector $v \in H$: $\Lambda(u) = \langle u, v\rangle$.

(3) Let $1 \le p < \infty$ and $q = p/(p-1)$ be a pair of conjugate exponents. Suppose that $f : [0,1] \to \mathbb{R}$ is a real valued function such that fg is integrable for all $g \in L^p[0,1]$. Show that $f \in L^q[0,1]$.

(4) Suppose that T is an everywhere defined symmetric linear operator on a Hilbert space H, $\langle Tx, y\rangle = \langle x, Ty\rangle$. Prove that T is a bounded operator.

(5) The following is a sequence of problems on $C[-1,1]$ and $L^\infty[-1,1]$.
   (a) Define the Banach spaces $C[-1,1]$ and $L^\infty[-1,1]$ where both spaces are equipped with the L^∞ norm (i.e. define this norm). Here $C[-1,1]$ is the space of continuous functions on $[-1,1]$.
   (b) Is $C[-1,1]$ a closed subspace of $L^\infty[-1,1]$? Prove that your answer is correct.
   (c) Let δ_0 be the point mass measure at 0. Show that $\langle \delta_0, f\rangle = f(0)$ defines a bounded linear functional on $C[-1,1]$. What is its norm?
   (d) Does δ_0 extend from $C[-1,1]$ to $L^\infty[-1,1]$ as a bounded linear functional? Explain your answer. You can cite relevant theorems but you do not need to prove them.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $f(z) = 1/(z^2 - 1)$.
   (a) Show that f has a well-defined analytic primitive on the slit plane $\mathbb{C} \setminus [-1,1]$.
   (b) Compute the integral $\int_\gamma f(z)\,dz$ along the path $\gamma(t) = 2e^{it}$ for $0 \le t \le \pi$.

(2) Provide an explicit description of the group of conformal automorphisms of the punctured disk $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

(3) Describe the following subsets of the complex plane:
   (a) $\left\{z : e^{2\pi z} = i\right\} \cap \left\{z : |z^3| \le 1000\right\}$;
   (b) $\left\{z : \mathrm{Im}\left(\frac{1}{i} \cdot \frac{z-3}{z+3}\right) > 0\right\}$;
   (c) the image of the vertical strip $\{z = x + iy : 0 < x < \pi\}$ under $f(z) = \cos z$.

(4) Fix an integer $n \ge 0$. Suppose f is analytic on an open set containing the closed unit disk $\{z : |z| \le 1\}$. Suppose further that $|f(z)| = 1$ for all $|z| = 1$ and that f has simple zeroes at a set of distinct points $\{a_1, \ldots, a_n\}$ in the disk. Find (and prove) a formula for f. **Hint:** consider first the cases where $n = 0$ and $n = 1$.

(5) Let U be an open, connected subset of $\mathbb{C}$. Prove the Weierstrass/Hurwitz Theorem: if f_n is a sequence of non-vanishing analytic functions on U converging uniformly on compact subsets of U to a function f, then f is either a non-vanishing analytic function or $f \equiv 0$.