PRELIMINARY EXAM IN ANALYSIS SPRING 2013

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) State and prove Fatou's Lemma (you may use the monotone convergence theorem).

(2) Show that, if $f : \mathbb{R} \to \mathbb{R}$ is measurable, then the set $\{x \in \mathbb{R} \mid m(f^{-1}(x)) > 0\}$ has measure zero.

(3) Let $A_n \subset [0,1]$ be subsets of measure $1/2$. Show that the set of points that are contained in infinitely many of the A_n's has measure at least $1/2$.

(4) Let $\{f_n\} \subset L^1([0,1], dx)$. Suppose that $f_n \to f$ a.e. on $[0,1]$. Show:

(a)
$$\int_0^1 |f_n| \to \int_0^1 |f| dx \implies ||f_n - f||_{L^1[0,1]} \to 0.$$

(b) Also construct an example where $f_n \to 0$ a.e. but $\int |f_n| dx$ does not converge to zero.

Hint: Think about $h_n(x) = \frac{|f(x)|+|f_n(x)|}{2} - \left|\frac{f_n(x)-f(x)}{2}\right|$.

(5) Let (Ω, μ) be a measure space and let $f \in L^\infty(X, \mu)$ be a positive measurable bounded function. Let ν be a measure on $[0, \infty]$ and let $\phi(t) = \nu[0, t)$. Consider the distribution function $\mu\{x \in X : f(x) > t\}$ of f. Prove the formulae:

(a) $\int_\Omega \phi(f(x)) d\mu(x) = \int_0^\infty \mu\{f > t\} d\nu(t)$.

(b) $f(x) = \int_0^\infty \chi_{\{y : f(y) > t\}}(x) dt$.

Here, χ_E is the characteristic (indicator function) of E.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $g \in L^1(\mathbb{R}, dx)$ satisfy $\int_{\mathbb{R}} g(x)dx = 1$. Calculate

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x)\sin^2(nx)dx$$

and prove that your answer is correct.

(2) Let $1 < p < \infty$, let $\{u_n\}_{n=1}^{\infty} \subset L^p(X,\mu)$, $\{v_n\}_{n=1}^{\infty} \subset L^q(X,\mu)$ be two sequences with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $u_n \to u$ weakly in L^p and $v_n \to v$ strongly in L^q. Prove

- $\{u_n\}$ is a bounded family in L^p.
- that $u_n v_n \to uv$ weakly in L^1.

(3) Suppose that $T, U \in \mathcal{L}(H)$ are bounded linear operators on a Hilbert space H, with U unitary. Suppose that $||T - U|| < 1$. Show that T is invertible. (Here, $||\cdot||$ is the operator norm.)

(4) Suppose that K is a self-adjoint compact operator on a Hilbert space H, and supose that K is positive in the sense that $\langle Kf, f\rangle \geq 0$ for all $f \in H$. Suppose that $Kf = g$ and $Kg = f$. Show that $f = g$.

(5) Let $C[0,1]$ be the continuous functions on $[01,]$ equipped with the sup norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Let $C^1[0,1] \subset C[0,1]$ be the C^1 functions (one continuous derivative), viewed as a subspace of $(C[0,1], ||\cdot|_\infty)$

(a) Show that $\frac{d}{dx} : (C^1[0,1], ||\cdot||_\infty \to C^0[0,1],)$ has a closed graph but is not a bounded linear operator.

(b) Why does this not contradict the closed graph theorem? Prove that your answer is correct.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Show that there exists a well-defined branch of the logarithm Log$z$ in the disc $|z-1-i| = \frac{5}{4}$ centered at $1+i$. Then calculate

$$\int_{|z-1-i|=\frac{5}{4}} \frac{\text{Log} z}{(z-1)^2} dz.$$

(2) Is the punctured plane $\mathbb{C}\backslash\{0\}$ conformally equivalent (i.e. biholomorphic) to the punctured unit disc $\mathbb{D}\backslash\{0\}$ ($\mathbb{D}$ is the open unit disc; i.e $\{z : 0 < |z| < 1\}$. Prove that your answer is correct.

(3) You may want to recall the hyperbolic pseudo-distance function on the unit disc: $\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|$.

  (a) Let $a, b \in \mathbb{D}$. Does there exist an automorphism $f : D \to D$ of the unit disc to itself satisfying $f(a) = b$ and and $f(b) = a$? Prove that your answer is correct.

  (b) Let $a_1, a_2, b_1, b_2 \in \mathbb{D}$. Give a necessary and sufficient condition for the existence of an automorphism g (a composition of a Moebius transformation $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ or a rotation $T_\theta = e^{i\theta}$) such that $g(a_1) = b_1, g(a_2) = b_2$?

(4) How many zeros does $z^9 + z^5 - 8z^3 + 2z + 1$ have between the circles $\{|z| = 1\}$ and $\{|z| = 2\}$

(5) Let f be analytic in the open unit disc $\mathbb{D}$ and suppose its zeros are $\{a_1, \ldots, a_n\}$, and that it has no zeros on $\partial\mathbb{D}$. Let $M = \sup_{z\in\mathbb{D}} |f(z)|$. Show that for $|z| < 1$,

$$|f(z)| \leq M \prod_{j=1}^{n} \frac{|z - a_j|}{|1 - \bar{a}_j z|}.$$