Preliminary Exam in Analysis Spring 2013

Instructions:
(1) There are three parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do three problems from each part.
(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory
Do three of the following five problems.

(1) State and prove Fatou’s Lemma (you may use the monotone convergence theorem).

(2) Show that, if $f : \mathbb{R} \to \mathbb{R}$ is measurable, then the set \( \{ x \in \mathbb{R} \mid m(f^{-1}(x)) > 0 \} \) has measure zero.

(3) Let \( A_n \subset [0,1] \) be subsets of measure 1/2. Show that the set of points that are contained in infinitely many of the \( A_n \)’s has measure at least 1/2.

(4) Let \( \{ f_n \} \subset L^1([0,1], dx) \). Suppose that \( f_n \to f \) a.e. on \([0,1]\). Show:
   (a) \( \int_0^1 |f_n| \to \int_0^1 |f| dx \implies \|f_n - f\|_{L^1[0,1]} \to 0. \)
   (b) Also construct an example where \( f_n \to 0 \) a.e. but \( \int |f_n| dx \) does not converge to zero.
   
   Hint: Think about \( h_n(x) = \frac{|f(x)| + |f_n(x)|}{2} - \frac{|f_n(x) - f(x)|}{2} \).

(5) Let \( (\Omega, \mu) \) be a measure space and let \( f \in L^\infty(X, \mu) \) be a positive measurable bounded function. Let \( v \) be a measure on \([0,\infty]\) and let \( \phi(t) = \nu[0,t) \). Consider the distribution function \( \mu\{ x \in X : f(x) > t \} \) of \( f \). Prove the formulae:
   (a) \( \int_{\Omega} \phi(f(x)) d\mu(x) = \int_0^\infty \mu\{ f > t \} dv(t). \)
   (b) \( f(x) = \int_0^\infty \chi_{\{ y : f(y) > t \}}(x) dt. \)

Here, \( \chi_E \) is the characteristic (indicator function) of \( E \).

Part II. Functional Analysis
Do three of the following five problems.
\( \textbf{(1)} \) Let \( g \in L^1(\mathbb{R}, dx) \) satisfy \( \int_{\mathbb{R}} g(x) dx = 1 \). Calculate
\[
\lim_{n \to \infty} \int_{\mathbb{R}} g(x) \sin^2(nx) dx
\]
and prove that your answer is correct.

\( \textbf{(2)} \) Let \( 1 < p < \infty \), let \( \{u_n\}_n \subset L^p(X, \mu) \), \( \{v_n\}_n \subset L^q(X, \mu) \) be two sequences with \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose that \( u_n \to u \) weakly in \( L^p \) and \( v_n \to v \) strongly in \( L^q \).

Prove
- \( \{u_n\} \) is a bounded family in \( L^p \).
- that \( u_nv_n \to uv \) weakly in \( L^1 \).

\( \textbf{(3)} \) Suppose that \( T, U \in L(H) \) are bounded linear operators on a Hilbert space \( H \), with \( U \) unitary. Suppose that \( \|T - U\| < 1 \). Show that \( T \) is invertible. (Here, \( \|\cdot\| \) is the operator norm.)

\( \textbf{(4)} \) Suppose that \( K \) is a self-adjoint compact operator on a Hilbert space \( H \), and suppose that \( K \) is positive in the sense that \( \langle Kf, f \rangle \geq 0 \) for all \( f \in H \). Suppose that \( Kf = g \) and \( Kg = f \). Show that \( f = g \).

\( \textbf{(5)} \) Let \( C[0,1] \) be the continuous functions on \([0,1]\) equipped with the sup norm \( \|f\|_{\infty} = \sup_{x \in [0,1]} |f(x)| \). Let \( C^1[0,1] \subset C[0,1] \) be the \( C^1 \) functions (one continuous derivative), viewed as a subspace of \( (C[0,1], \|\cdot\|_{\infty}) \)

(a) Show that \( \frac{d}{dx} : (C^1[0,1], \|\cdot\|_{\infty}) \to C^0[0,1], \) has a closed graph but is not a bounded linear operator.

(b) Why does this not contradict the closed graph theorem? Prove that your answer is correct.

**Part III. Complex Analysis**

Do three of the following five problems.

\( \textbf{(1)} \) Show that there exists a well-defined branch of the logarithm \( \text{Log} z \) in the disc \( |z - 1 - i| = \frac{\pi}{4} \) centered at \( 1 + i \). Then calculate
\[
\int_{|z-1-i|=\frac{\pi}{4}} \frac{\text{Log} z}{(z-1)^2} dz.
\]

\( \textbf{(2)} \) Is the punctured plane \( \mathbb{C} \setminus \{0\} \) conformally equivalent (i.e. biholomorphic) to the punctured unit disc \( \mathbb{D} \setminus \{0\} \) (\( \mathbb{D} \) is the open unit disc; i.e \( \{ z : 0 < |z| < 1 \} \). Prove that your answer is correct.
(3) You may want to recall the hyperbolic pseudo-distance function on the unit disc:
\[ \rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|. \]

(a) Let \( a, b \in D \). Does there exist an automorphism \( f : D \to D \) of the unit disc to itself satisfying \( f(a) = b \) and \( f(b) = a \)? Prove that your answer is correct.

(b) Let \( a_1, a_2, b_1, b_2 \in D \). Give a necessary and sufficient condition for the existence of an automorphism \( g \) (a composition of a Moebius transformation \( B_{a}(z) = \frac{z-a}{1-\overline{a}z} \) or a rotation \( T_{\theta} = e^{i\theta} \)) such that \( g(a_1) = b_1, g(a_2) = b_2 \)?

(4) How many zeros does \( z^9 + z^5 - 8z^3 + 2z + 1 \) have between the circles \( \{ |z| = 1 \} \) and \( \{ |z| = 2 \} \)

(5) Let \( f \) be analytic in the open unit disc \( D \) and suppose its zeros are \( \{ a_1, \ldots, a_n \} \), and that it has no zeros on \( \partial D \). Let \( M = \sup_{z \in D} |f(z)| \). Show that for \( |z| < 1 \),
\[ |f(z)| \leq M \prod_{j=1}^{n} \frac{|z - a_j|}{|1 - \overline{a_j}z|}. \]