Preliminary Examination for Real and Complex Analysis-September 2002

Part I

Do all three problems (1)-(2)-(3) in this section.

(1a) Define the concepts of measure space and real measurable function.

(1b) Let (Ω, Σ, μ) be a measure space. Show that the following property holds. All sets considered are members of Σ.

$$\text{If } \mu(A_1) < \infty,\ A_1 \supset A_2 \supset A_3 \supset \dots, \text{then } \lim_{j\to\infty} \mu(A_j) = \mu(\cap_{i=1}^{\infty} A_i).$$

(1c) Let μ be a measure defined on an algebra $\mathcal{A}$ of subsets of Ω. Describe how an outer measure μ^* can be constructed on all subsets of Ω. How can a measure ν be defined from μ^*? What is the relation of this measure to μ?

(1d) Let Ω be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(\Omega)$. How does one define an outer measure on the subsets of Ω?

(2) State and prove the Riemann-Lebesgue lemma.

(3) Suppose that (f_n) is a uniformly bounded sequence of holomorphic functions in the unit disk U such that $f(z) := \lim_n f_n(z)$ exists for each $z \in U$. Prove that the convergence is uniform on $\{z : |z| \le r\}$ for each $r < 1$ and that f is holomorphic in U.

Part II

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) State and prove Jensen's inequality.

(2) Let X be a real Banach space, and suppose that C_1 is an open convex set and C_2 is convex, with $C_1 \cap C_2 = \emptyset$. Show that there is a separating hyperplane, i. e., show that there exists a real linear functional $T \in X^*$ and a real number α such that

$$Tx < \alpha \le Ty,\ x \in C_1,\ y \in C_2.$$

Part III

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) The Banach-Alaoglu theorem for $L^p(\Omega), \Omega \subset \mathbb{R}^n$ with Lebesgue measure, and $1 < p < \infty$, asserts that every bounded sequence in $L^p(\Omega)$ has a weakly convergent subsequence. Give the proof.

(2) If $\mu(\Omega) < \infty$, a sufficient condition for a function $f \in L^1(\Omega)$ to have range up to a set of measure zero in a closed set $S \subset \mathbb{C}$ is that the averages,

$$\frac{1}{\mu(E)} \int_E f \, d\mu,$$

lie in S for each $E \in \Sigma$ with $\mu(E) > 0$. Prove this.

Part IV

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) Suppose that $f = u + iv$ is holomorphic in the unit disk with $f(0) = 0$. Prove that for each $k = 1, 2, \ldots$ there is a constant C_k such that

$$\int_0^{2\pi} |v(re^{i\theta})|^{2k} \, d\theta \leq C_k \int_0^{2\pi} |u(re^{i\theta})|^{2k} \, d\theta, \qquad \forall r < 1.$$

(2) Suppose that f is holomorphic in the unit disk U with $|f(z)| < 1$ and that f has two distinct fixed points z_1, z_2, satisfying $f(z_1) = z_1 \neq z_2 = f(z_2)$. Prove that $f(z) = z$ for all $z \in U$.