Preliminary Examination for
Real and Complex Analysis-September 2002

Part I

Do all three problems (1)-(2)-(3) in this section.

(1a) Define the concepts of measure space and real measurable function.

(1b) Let \((\Omega, \Sigma, \mu)\) be a measure space. Show that the following property holds. All sets considered are members of \(\Sigma\).

\[
\text{If } \mu(A_1) < \infty, \ A_1 \supset A_2 \supset A_3 \supset \ldots, \text{ then } \lim_{j \to \infty} \mu(A_j) = \mu(\bigcap_{i=1}^{\infty} A_i).
\]

(1c) Let \(\mu\) be a measure defined on an algebra \(\mathcal{A}\) of subsets of \(\Omega\). Describe how an outer measure \(\mu^*\) can be constructed on all subsets of \(\Omega\). How can a measure \(\nu\) be defined from \(\mu^*\)? What is the relation of this measure to \(\mu\)?

(1d) Let \(\Omega\) be a locally compact Hausdorff space, and let \(\Lambda\) be a positive linear functional on \(C_c(\Omega)\). How does one define an outer measure on the subsets of \(\Omega\)?

(2) State and prove the Riemann-Lebesgue lemma.

(3) Suppose that \((f_n)\) is a uniformly bounded sequence of holomorphic functions in the unit disk \(U\) such that \(f(z) := \lim_n f_n(z)\) exists for each \(z \in U\). Prove that the convergence is uniform on \(\{z : |z| \leq r\}\) for each \(r < 1\) and that \(f\) is holomorphic in \(U\).

Part II

Do either but not both of the problems in this section. Otherwise, only the first problem will be graded.

(1) State and prove Jensen’s inequality.

(2) Let \(X\) be a real Banach space, and suppose that \(C_1\) is an open convex set and \(C_2\) is convex, with \(C_1 \cap C_2 = \emptyset\). Show that there is a separating hyperplane, i.e., show that there exists a real linear functional \(T \in X^*\) and a real number \(\alpha\) such that

\[
Tx < \alpha \leq Ty, \ x \in C_1, \ y \in C_2.
\]
**Part III**

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) The Banach-Alaoglu theorem for $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ with Lebesgue measure, and $1 < p < \infty$, asserts that every bounded sequence in $L^p(\Omega)$ has a weakly convergent subsequence. Give the proof.

(2) If $\mu(\Omega) < \infty$, a sufficient condition for a function $f \in L^1(\Omega)$ to have range up to a set of measure zero in a closed set $S \subset \mathbb{C}$ is that the averages,

$$\frac{1}{\mu(E)} \int_E f \, d\mu,$$

lie in $S$ for each $E \in \Sigma$ with $\mu(E) > 0$. Prove this.

**Part IV**

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) Suppose that $f = u + iv$ is holomorphic in the unit disk with $f(0) = 0$. Prove that for each $k = 1, 2, \ldots$ there is a constant $C_k$ such that

$$\int_0^{2\pi} |v(re^{i\theta})|^{2k} \, d\theta \leq C_k \int_0^{2\pi} |u(re^{i\theta})|^{2k} \, d\theta, \quad \forall r < 1.$$

(2) Suppose that $f$ is holomorphic in the unit disk $U$ with $|f(z)| < 1$ and that $f$ has two distinct fixed points $z_1, z_2$, satisfying $f(z_1) = z_1 \neq z_2 = f(z_2)$. Prove that $f(z) = z$ for all $z \in U$.  

2