Preliminary Examination for Real and Complex Analysis-September, 2003

Part I

Do all three problems (1)-(2)-(3) in this section.

(1a) Let $\mathcal{A}$ be an algebra of subsets of Ω and let μ be a measure on $\mathcal{A}$. State the properties of an outer measure and explain how an outer measure is constructed on all subsets of Ω from μ. How are measurable sets defined? Explain how Lebesgue measure is constructed as a special case of this process.

(1b) Show that the measure space of part (1a) is complete.

(1c) How is a product measure space defined from two measure spaces? Discuss both the product sigma algebra and the product measure.

(2) State and prove the Lebesgue Dominated Convergence Theorem. You may use Fatou's lemma.

(3) Suppose that (f_n) is a uniformly bounded sequence of holomorphic functions in the unit disk U such that $f(z) := \lim_n f_n(z)$ exists for each $z \in U$. Prove that the convergence is uniform on $\{z : |z| \leq r\}$ for each $r < 1$ and that f is holomorphic in U.

Part II

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) Find a counter-example to the complex Stone-Weierstrass theorem if the closure under complex conjugation is not included as an hypothesis.

(2) Prove Egoroff's theorem:

Let (Ω, Σ, μ) be a measure space, with $\mu(\Omega) < \infty$. Let $\{f^j\}$ be a sequence of complex-valued measurable functions on Ω which is pointwise convergent μ-a.e. to a function f. For each $\epsilon > 0$, there is a set $A \in \Sigma$ such that $\mu(A^c) < \epsilon$ and $\{f^j\}$ converges uniformly to f on A.

Part III

Do *either* but not *both* of the problems in this section. Otherwise, only the first problem will be graded.

(1) State and prove the Hahn-Banach theorem.

(2) Assume that $1 \leq p < \infty$. Under the assumption that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, prove that $L^p(\mathbb{R}^n)$ is separable.

Part IV

Do the single problem in this section.

(1) Let $\Omega := \{z : |z| \neq 1\}$, $f \in L^1(T)$, $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$ and consider the Cauchy integral

$$u(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi, \qquad z \in \Omega.$$

(a) Prove that $u \in H(\Omega)$ with $\lim_{z \to \infty} u(z) = 0$.

(b) If $f \in C(T)$, prove that $\lim_{z \to e^{i\theta}, |z|<1} [u(z) - u(1/\bar{z})] = f(e^{i\theta})$, uniformly on T.

(c) If $f \in L^p(T)$ for some $1 \leq p < \infty$, prove that the limit in part (b) takes place in $L^p(T)$.