Preliminary Examination for
Real and Complex Analysis-September, 2004

There are 300 total points. The problems in Part I are worth 20 points each, while the problems in Parts II and III are worth 40 points.

Part I

Do all three problems in this section.

(1) Given \((\Omega, \Sigma)\), where \(\Sigma\) is a sigma-algebra of subsets of \(\Omega\), define a real-valued measurable function, and prove that the sum of two such functions is measurable.

(2) Suppose a real-valued nonnegative function \(f\) is summable on the measure space \((\Omega, \Sigma, \mu)\) and

\[
\int_{\Omega} f \, d\mu = 0.
\]

What do you conclude about \(f\)? Give the proof.

(3) Find the Laurent expansion for the function

\[
f(z) = \frac{z - 1}{z(z - 2)^3}
\]

in the annulus \(\{z : 0 < |z - 2| < 2\}\).
Part II

Do any four of the problems in this section. If you do more than four, indicate which ones should be graded.

(1) Prove the following theorem.
   Theorem. A normed linear space \( X \) is complete if, whenever \( \{x_n\} \subset X \) and \( \sum_{n=1}^{\infty} \|x_n\| < \infty \), then \( \sum_{n=1}^{\infty} x_n \in X \), that is, if every absolutely convergent series is convergent.
   Use this theorem directly to prove the completeness of \( L^p(\Omega) \), \( 1 \leq p < \infty \).

(2) Define the translation operator
   \[
   (\tau_h f)(x) = f(x - h), \quad h, x \in \mathbb{R}^n.
   \]
   Prove that \( \tau_h \) is continuous on \( L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \). State explicitly what property of the continuous compact support functions and what property of Lebesgue measure you are using in the course of the proof.

(3) Give a counter-example to show that a closed operator need not be continuous. Then state and prove the closed graph theorem.

(4) Let \( (\Omega, \Sigma, \mu) \) be a measure space. Show that this space has a completion \( (\Omega, \tilde{\Sigma}, \tilde{\mu}) \), defined as follows.
   \[
   \tilde{\Sigma} = \{E \cup A : E \in \Sigma, \quad A \subset B \text{ for some } B \in \Sigma \text{ such that } \mu(B) = 0\},
   \]
   \[
   \tilde{\mu}(E \cup A) = \mu(E).
   \]

(5) Let \( X \) be a complete metric space. Prove that any countable collection of dense open subsets of \( X \) has nonempty intersection.

(6) Determine the Fourier transform of
   \[
   g(x) = \exp(-\pi|x|^2), \quad x \in \mathbb{R}^n.
   \]
Part III

Do any two problems in this section. If you do more than two, indicate which ones should be graded.

(1) Find all linear fractional transformations $\phi$ that map the upper half-plane onto the disk $D = \{w : |w| < R\}$.

(2) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1 + a \sin \theta}, \quad -1 < a < 1.$$ 

(3) How many zeros does $\sin(z) + 2iz^2$ have inside the rectangle,

$$\{z : \text{Re}(z) < \frac{\pi}{2}, \text{Im}(z) \leq 1\}?$$

Justify your answer.