Real and Complex Analysis

PART A. Do problems 1, 2 and 3

1. (a) Define
   
   \[ f(t) = \int_0^{100} \frac{\cos(1/x + t)}{|x+t|^{1/2}} \, dx. \]

   Prove or disprove: \( \lim_{t \to 0^+} f(t) = f(0) \).

   (b) Define
   
   \[ g(t) = \int_0^{100} \frac{\cos^{-1/2}(x)}{|x+t^{1/2}|} \, dx. \]

   Prove or disprove: \( \lim_{t \to 0^+} g(t) = g(0) \).

   The notation \( t \to 0^+ \) indicates limit from the right (the positive \( t \)-axis).

2. Prove or give a counterexample for each of the following statements. The functions \( f, f_1, f_2, \ldots \) are assumed to be measurable.

   (a) The unit ball in \( L^1([0,1]) \) is compact.

   (b) If \( f_i \to f \) in \( L^2(\mathbb{R}^n) \), then \( f_i \to f \) a.e.

   (c) Suppose \( f, f_1, f_2, \ldots \) possess a common bound in \( L^2(\mathbb{R}^n) \) and in \( L^\infty(\mathbb{R}^n) \). If \( f_i \to f \) a.e., then \( f_i \to f \) in \( L^2(\mathbb{R}^n) \).

   (d) Suppose \( f, f_1, f_2, \ldots \) possess a common \( L^\infty \) bound in \( B_1 \), the unit ball in \( \mathbb{R}^n \). If \( f_i \to f \) in measure in \( B_1 \), then \( f_i \to f \) in \( L^2(B_1) \).

3. Use the method of residues to evaluate the integral

   \[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} \, dx. \]

PART B. Do problem 4 or 5

4. Suppose that \( h, h_1, h_2, \ldots \) are \( L^2 \) functions on \( \mathbb{R}^n \) such that for every continuous function \( \phi \) of compact support,

   \[ \lim_{i \to \infty} \int_{\mathbb{R}^n} \phi(x)h_i(x) \, dx = \int_{\mathbb{R}^n} \phi(x)h(x) \, dx, \]

   Prove that

   \[ \int_{\mathbb{R}^n} |h(x)|^2 \, dx \leq \lim_{i \to \infty} \int_{\mathbb{R}^n} |h_i(x)|^2 \, dx. \]

5. Prove that if \( u \) is a function in \( L^2([0,2\pi]) \), then

   \[ \lim_{n \to \infty} \int_{0}^{2\pi} u(x) \sin(nx) \, dx = 0. \]
PART C.  Do problem 6 or 7

6. Let $B_1$ be the ball of radius 1 in $\mathbb{R}^n$. Suppose that $f$ is simultaneously in $L^p(B_1)$ for each $p \in [1, \infty]$. Prove that

$$\lim_{p \to \infty} \|f\|_{L^p(B_1)} = \|f\|_{L^\infty(B_1)}.$$ 

7. Let $f$ be an $L^1$ function on $\mathbb{R}^n$. Prove that for a.e. $x$ in $\mathbb{R}^n$,

$$\lim_{r \to \infty} \frac{1}{|B_r|} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0,$$

where $B_r(x)$ is the ball in $\mathbb{R}^n$ with center $x$ and radius $r$, and $B_r = B_r(0)$. (You are allowed to assume the weak $L^1$ estimate on the maximal function.)

PART D.  Do problem 8 or 9

8. How many zeros does $z^6 + 4z^2 - 1$ have in the annulus $\{z : 1 \leq |z| \leq 2\}$?

9. (a) Find a linear fractional transformation that maps 0 to $i$, 1 to $\infty$, and $-1$ to 1.

(b) Suppose $f(z) \in H(\Omega)$, $\Omega$ contains the closed unit disc, $f(0) = 1/2$, and $|f(z)| > 1$ when $|z| = 1$. Prove that $f(z)$ has a zero in the unit disc.