Preliminary Examination – Real and Complex Analysis – Sept. 14, 1998

Instructions: You have **three** hours. Write your I.D. number on each of your bluebooks. Do **two** problems in each of the three parts — **six** problems in all. If you try more than two problems in any part, indicate which of the problems are to be graded. You may assume that all functions are real-valued in PART I and PART II. In PART III, we denote $B(0,1) = \{|z| < 1\}$. **Justify your statements.**

PART I

1. Define the Cantor ternary function. Prove that it is continuous and of bounded variation, but **not** absolutely continuous.

2. State and prove Egoroff's theorem. Find a sequence of functions which shows that the restriction to spaces of finite measure in Egoroff's theorem is essential.

3. Prove that the space c of all convergent sequences of real numbers is a Banach space (with the ℓ^∞ norm).

PART II

4. Prove that the dual (or conjugate) of L^2 is L^2.

5. State the open mapping theorem and the closed graph theorem. Prove that if X and Y are Banach spaces and Λ is a one-to-one bounded linear transformation of X onto Y, then Λ^{-1} is a bounded linear transformation.

6. State the Baire category theorem and use it to prove that the set of irrational numbers is **not** an F_σ set.

PART III

7. Evaluate the following definite integrals:

(a) $\int_0^\infty \left(\frac{sin(\alpha x)}{x}\right)^2 dx, \quad \alpha \neq 0$ is real.

(b) $\int_0^\infty \frac{x^\alpha}{x^2+5x+4}\, dx, \quad 0 < \alpha < 1.$

8. Let f be analytic on $B(0,1)$ with $f(z) = \sum a_n z^n$ and $|f(z)| \leq \frac{1+|z|}{1-|z|}$. Prove that $|a_n| \leq (2n+1)\left(1+\frac{1}{n}\right)^n$ for $n = 1, 2, \ldots$.

9. Prove that any conformal and bijective map $f(z)$ of $B(0,1)$ onto itself, satisfying $f(a) = 0$ for some a with $|a| < 1$, must be of the form

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

for some fixed $\theta \in [0, 2\pi)$.