Preliminary Examination — Real Analysis — Feb. 19. 1998

Instructions: You have **three** hours. Write your I.D. number on two bluebooks and mark them A and B. Do **six** of the eight problems below — **three** problems from Part A in bluebook A, and **three** problems from Part B in bluebook B. Indicate which of the problems are to be graded. You may assume that all functions are real-valued.

PART A

A1. Prove that if $f \in L^1[a,b]$ and

$$F(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$

then $F'(x) = f(x)$ for almost every x in $[a,b]$.

A2. Let $(X, \mathcal{B}, \mu)$ be a complete measure space, $g \in L^q(\mu)$, $1 \le q < \infty$, and let F be the linear functional on $L^p(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, defined by $F(f) = \int fg\,d\mu$. Prove that $\|F\| = \|g\|_q$.

A3. Prove that the space c_o of all sequences which converge to zero is a Banach space (with the ℓ^∞ norm).

A4. State Fubini's Theorem. Prove that if $f \in L^1(0,1)$ and $a > 0$, then the integral

$$F_a(x) = \int_0^x (x-t)^{a-1} f(t)\,dt$$

exists for almost every x in $(0,1)$ and $F_a \in L^1(0,1)$.

PART B

B1. Prove that a function f is of bounded variation on a compact interval $[a,b]$ if and only if f is the difference of two monotonic functions on $[a,b]$.

B2. State the Hahn Decomposition Theorem. Prove that if ν is a signed measure on a measurable space $(X, \mathcal{B})$, then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{B})$ such that $\nu = \nu^+ - \nu^-$. Also prove that there is only one such pair of mutually singular measures.

B3. State the Radon-Nikodym Theorem. Prove that if $(X, \mathcal{B}, \mu)$ is a σ-finite measure space and ν is a σ-finite measure defined on $\mathcal{B}$, then there exists a measure ν_0, singular with respect to μ, and a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$.

B4. State the Hahn-Banach Theorem. Prove that if x_o is an element in a normed vector space X, then there is a bounded linear functional f on X such that $f(x_o) = \|f\|\|x_o\|$.