GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, SPRING 2016

Answer all six questions.

(1) For this question, you may make use of any standard results, so long as they are referenced explicitly.
  (a) Identify the Klein bottle K as quotient of a polygon.
  (b) Compute the fundamental group
  $$\pi_1 K$$
  in terms of generators and relations, and prove your answer.
  (c) Construct a double covering map $\mathbb{T}^2 \to K$ from the torus to the Klein bottle.
  (d) Calculate the Euler characteristic $\chi(K)$ of the Klein bottle.
  (e) Compute the induced map on fundamental groups
  $$\pi_1 \mathbb{T}^2 \to \pi_1 K$$
  in terms of your presentation of $\pi_1 K$.

(2) Consider the map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$f(x, y) = (x^2 - y^2, 2xy).$$
  (a) Compute f_* and find the maximal domain X on which f is a submersion.
  (b) Let g be the Euclidean metric on the codomain $\mathbb{R}^2$ (the image of f). Compute the pull-back metric $f^*(g)$ on your X from part (a). Show that it is conformally Euclidean, meaning the angles between two vectors are the same as for the Euclidean metric (The cosine of the angle between two vectors v, w, is defined as $\frac{\langle v,w\rangle}{|v|\,|w|}$.)
  (c) Find a geodesic parametrized by arclength on X through the point $(1, 0)$ in the direction of $(1, 0)$, i.e. ∂_x.

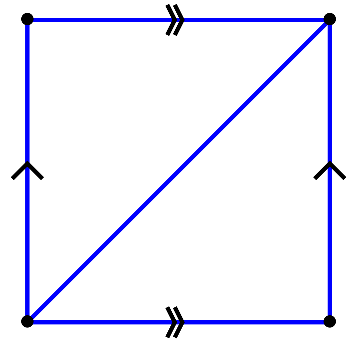
(3) Consider the projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, where $t \in \mathbb{R}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Every form in $\Omega^*(\mathbb{R}^n \times \mathbb{R})$ is a linear combination of forms like $f(x, t)\pi^*\theta(x)$ and $f(x, t)\pi^*\theta(x)dt$. Define a linear operator
$$H : \Omega^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{k-1}(\mathbb{R}^n \times \mathbb{R})$$
that sends forms of the first kind to zero and of the second kind to $\left(\int_0^t f(x, t)dt\right) \pi^*\theta(x)$.

  (a) Using H, prove the equivalence of de Rham cohomology $H^*_{dR}(\mathbb{R}^{n+1}) \cong H^*_{dR}(\mathbb{R}^n)$.

  Hint: Compute $dH - Hd$.

  (b) Now let $L \to M$ be any line bundle. Prove $H^*_{dR}(L) \cong H^*_{dR}(M)$.

(4) Use the Δ-complex structure for the torus T^2 indicated by this picture:



Let α be the simplicial cochain which evaluates to 1 on the left edge and 0 on all other edges. (You will need to decide how to label/order vertices to make this precise.) Let β be the cochain which evaluates to 1 on the top edge and 0 on all other edges.

Let δ be the differential acting on simplicial cochains. Compute the following:

$$\delta\alpha, \qquad \delta\beta, \qquad \alpha \cup \beta.$$

(5) For this question, you may make use of any standard results, so long as they are referenced explicitly. A k-flag $V_\bullet$ in $\mathbb{C}^n$ is a sequence of inclusions of complex vector spaces

$$V_\bullet = \left\{V_1 \subset V_2 \subset \ldots \subset V_k \subset \mathbb{C}^n\right\}$$

for which $\mathsf{dim}_{\mathbb{C}}(V_i) = i$ for $1 \leq i \leq k$.

(a) Prove that the collection $\mathsf{Fl}_k(\mathbb{C}^n)$ of k-flags in $\mathbb{C}^n$ can be identified with the quotient

$$\mathsf{U}(n)\big/(\mathbb{T}^k \times \mathsf{U}(n-k))$$

of the unitary group by the product of torus and a unitary group. Conclude $\mathsf{Fl}_k(\mathbb{C}^n)$ has the structure of a smooth manifold.

(b) Define a map

$$\mathsf{Fl}_k\big(\mathbb{C}^n\big) \to \mathsf{Fl}_{k-1}\big(\mathbb{C}^n\big)$$

and prove that this map is a smooth fiber bundle. Identify the fiber of this map over a $(k-1)$-flag $V_\bullet \in \mathsf{Fl}_{k-1}(\mathbb{C}^n)$.

(c) Calculate the Euler characteristic

$$\chi\big(\mathsf{Fl}_k(\mathbb{C}^3)\big)$$

for $k \in \{1, 2, 3\}$.

(6) Recall that $\mathbb{CP}^1$ is the space of complex lines through the origin in two complex dimensions. As a real manifold, it is two-dimensional. The tautological bundle $E \to \mathbb{CP}^1$ is the complex line bundle whose fiber over a point p is the line that p represents.

(a) Give a local trivialization for E and write down the transition maps.

(b) Give the closed Čech one-cocycle which classifies E up to isomorphism.

(c) Show that the one-cocycle for a new trivialization (in the same cover) lies in the same the cohomology class.

Remark: You do not have to worry about whether or not your cover is "good."