*Preliminary Examination* *September, 1993*

## Differential Geometry

*Do Six of the following problems*

**Problem 1.** (Space curves)

(a) Define the curvature of a space curve;

(b) Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a nondegenerate space curve. Show that the curvature at $t$ is given by

$$\kappa(t) = \frac{|\mathbf{r}_t \times \mathbf{r}_{tt}|}{|\mathbf{r}_t|^3}.$$

**Problem 2.** (General manifolds) Define the following terms:

(a) Differentiable manifold;

(b) Tangent bundle and vector field;

(c) Parallel translation;

(d) Covariant differentiation.

**Problem 3.** (Gaussian and mean curvatures)

(a) The Poincaré metric on a unit disk $D = \{(x, y) \in R^2 : x^2 + y^2 < 1\}$ is given by the formula

$$ds^2 = \frac{4\,(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}.$$

Compute the Gaussian curvature of $D$ with this metric;

(b) The *helicoid* $H$ can be parametrized as

$$(u, v) \longrightarrow (v\cos u, -v\sin u, bu), \quad (u, v) \in R^2.$$

($b$ is a nonnegative constant). Compute the first and second fundamental forms and the mean curvature of $H$.

**Problem 4.** (Lie groups) Let $G$ be a Lie group.

(a) Show that there is a unique connection $\nabla$ such that for all left-invariant vector fields $X$ and $Y$ we have $\nabla_X Y = [X, Y]$;

(b) Compute the torsion and curvature tensors of the connection $\nabla$;

(c) Show that under the connection given in (a) every one-parameter subgroup of $G$ is a geodesic, i.e., its tangent vector field is parallel along the curve;

(d) Can you think of other natural connections of $G$ with the property stated in (c)?.

**Problem 5.** (Polar coordinates) Let $S$ be a two dimensional manifold.

(a) Define geodesic polar coordinates $(r, \theta)$ in a neighborhood of a point $o \in M$;

(b) (Gauss' lemma) Show that in the polar coordinates the Riemannian metric can be written as

$$ds^2 = dr^2 + G(r, \theta)^2 d\theta^2,$$

where $G(r, \theta)$ is a smooth function on $[0, c] \times S^1$ such that $G(0, \theta) = 0$ and $G_r(0, \theta) = 1$ ($S^1$ stands for the unit circle);

(c) Show that the gaussian curvature at $(r, \theta)$ is given by

$$K(r, \theta) = -\frac{G_{rr}(r, \theta)}{G(r, \theta)};$$

(d) Show that if the gaussian curvature is identically equal to zero in a neighborhood of $o$, then $S$ is locally isometric to $R^2$ at $o$.

**Problem 6.** (Gauss-Bonnet theorem)

(a) State a version of Gauss-Bonnet theorem;

(b) Show that for any geodesic triangle on the unit sphere $S$ in $R^3$ the sum of the three angles is greater than $\pi$.

**Problem 7.** (Differential forms)

(a) State definitions of differential forms, closed forms, and exact forms;

(b) State a version of Stokes' theorem for differential forms on manifolds;

(c) Show that on $R^2$ every closed 1-form is exact;

(d) Show that on the 2-sphere $S^2$, every closed 1-form is exact.

**Problem 8.** (Riemannian connection) Let $M$ be a Riemannian manifold.

(a) If $\nabla$ and $\tilde{\nabla}$ are two connections on $M$, and define for two vector fields $X$ and $Y$,

$$S(X, Y) = \nabla_X Y - \tilde{\nabla}_X Y.$$

Show that $S$ is a tensor of type (1,2);

(b) Show that there is at most one connection on $M$ which is compatible with the Riemannian metric and is torsion-free.