Geometry Prelim Exam
Sep. 16, 1998

Work SIX of the following problems INCLUDING AT LEAST ONE from each of the groups 1-3, 4-6, 7-9, 10-12. Please list the six you want graded on the front of your blue book

1. Define chart, atlas and manifold. Define $C^\infty$ function from one manifold to another and define the derivative of such a map. Give an example of a $C^\infty$ homeomorphism whose inverse fails to have a derivative at some point.

2. Define submanifold, immersion and embedding. Give an example of an immersion of a manifold which is not an embedding.

3. (a) Define transversal intersection of two submanifolds.
   (b) Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and let $W$ be the graph of $A$, i.e. $W = \{(v, A(v)) : v \in \mathbb{R}^n\}$. Let $V = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Prove that $V$ is transverse to $W$ in $\mathbb{R}^n \times \mathbb{R}^n$ if and only if $A$ is an isomorphism.

4. (a) Define homotopy of two continuous maps.
   (b) Suppose $f(t)$ is a loop in $(X, x_0)$, i.e. a continuous function $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$. Let $g : [0, 1] \to [0, 1]$ be a continuous function satisfying $g(0) = 0$ and $g(1) = 1$. Prove that $f(g(t))$ is another loop in $(X, x_0)$, and that it is homotopic to the loop $f(t)$ relative to the base point $x_0$.

5. (a) Define critical point and critical value of a smooth function. State Sard’s Theorem. (b) Prove that if $\dim M < \dim N$ and $f : M \to N$ is $C^\infty$ then $f$ is not onto. In particular there are no $C^\infty$ space filling curves.

6. (a) Define differential $k$-form.
   (b) Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative. Prove that $d \circ d$ is identically zero.
   (c) Give an example of a one-form $\omega$ on $\mathbb{R}^4$ such that $d\omega \wedge d\omega$ is non-zero.

7. (a) Define affine connection, symmetric affine connection and affine connection compatible with a metric.
   (b) Let $\nabla_X Y = (X(g_1), X(g_2), X(g_3))$ if $Y = (g_1, g_2, g_3)$ is a vector field on $\mathbb{R}^3$. Prove that $\nabla$ is a symmetric affine connection compatible with the standard metric on $\mathbb{R}^3$.

8. (a) Define the Weingarten map for a surface $M^2$ in $\mathbb{R}^3$.
   (b) Consider the surface $z = 2x^2 - y^2$ in $\mathbb{R}^3$. Find the Gaussian curvature, the principal curvatures and the principal directions at the point $(0, 0, 0)$ on this surface.

9. (a) State the first and second Cartan Structure equations. (b) Prove one of them.