Geometry Prelim Exam
Sep. 16, 1998

Work **SIX** of the following problems *INCLUDING AT LEAST ONE* from each of the groups 1-3, 4-6, 7-9, 10-12. Please list the six you want graded on the front of your blue book

1. Define *chart, atlas* and *manifold.* Define C^∞ *function* from one manifold to another and define the *derivative* of such a map. Give an example of a C^∞ homeomorphism whose inverse fails to have a derivative at some point.

2. Define *submanifold, immersion* and *embedding.* Give an example of an immersion of a manifold which is not an embedding.

3. (a) Define *transversal intersection* of two submanifolds.
(b) Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and let W be the graph of A, i.e. $W = \{(v, A(v)) \in \mathbb{R}^n \times \mathbb{R}^n \mid v \in \mathbb{R}^n\}$. Let $V = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Prove that V is transverse to W in $\mathbb{R}^n \times \mathbb{R}^n$ if and only if A is an isomorphism.

4. (a) Define homotopy of two continuous maps.
(b) Suppose $f(t)$ is a loop in (X, x_0), i.e. a continuous function $f : [0,1] \to X$ with $f(0) = f(1) = x_0$. Let $g : [0,1] \to [0,1]$ be a continuous function satisfying $g(0) = 0$ and $g(1) = 1$. Prove that $f(g(t))$ is another loop in (X, x_0), and that it is homotopic to the loop $f(t)$ relative to the base point x_0.

5. (a) Define *critical point* and *critical value* of a smooth function. State Sard's Theorem. (b) Prove that if $dimM < dimN$ and $f : M \to N$ is C^∞ then f is not onto. In particular there are no C^∞ space filling curves.

6. (a) Define differential *k-form.*
(b) Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative. Prove that $d \circ d$ is identically zero.
(c) Give an example of a one-form ω on $\mathbb{R}^4$ such that $d\omega \wedge d\omega$ is non-zero.

7. (a) Define *affine connection, symmetric affine connection* and *affine connection compatible with a metric.*
(b) Let $\bar{\nabla}_X Y = (X(y_1), X(y_2), X(y_3))$ if $Y = (y_1, y_2, y_3)$ is a vector field on $\mathbb{R}^3$. Prove that $\bar{\nabla}$ is a symmetric affine connection compatible with the standard metric on $\mathbb{R}^3$.

8. (a) Define the *Weingarten map* for a surface M^2 in $\mathbb{R}^3$.
(b) Consider the surface $z = 2x^2 - y^2$ in $\mathbb{R}^3$. Find the Gaussian curvature, the principal curvatures and the principal directions at the point $(0,0,0)$ on this surface.

9. (a) State the first and second Cartan Structure equations. (b) Prove one of them.

OVER