Preliminary Exam - Differential Geometry (D41/D43, Spring 99/00)

Do all problems.

**Problem 1.** (a) Define the universal cover of connected differentiable manifold. (b) Show that the universal cover of the unit circle \( \{ e^{i\theta} : 0 \leq \theta < 2\pi \} \) is the real line \( \mathbb{R} = (-\infty, \infty) \).

**Problem 2.** (a) Define the orientability of a manifold. (b) Show that if a differentiable manifold \( M \) is covered by two charts \( U \) and \( V \) whose intersection \( U \cap V \) is connected, then \( M \) is orientable.

**Problem 3.** (a) Define the tangent space \( T_x M \) of a differentiable manifold \( M \) at a point \( x \in M \). (b) Show that \( T_x M \) has the same dimension as \( M \).

**Problem 4.** (a) Define the torsion and curvature of a connection. (b) Define the Christoffel symbols \( \Gamma^k_{ij} \) of a connection. (c) Show that the connection is torsion-free if and only if \( \Gamma^k_{ij} = \Gamma^k_{ji} \).

**Problem 5.** (a) Define the exterior derivative of a differential form, either invariantly or in local coordinates. (b) State Stoke’s theorem.

**Problem 6.** Let \( M \) be a noncompact Riemannian manifold and \( \{ O_n \} \) a sequence of relative compact open subset of \( M \) such that \( \overline{O}_n \subseteq O_{n+1} \) and \( \cup_{n=1}^{\infty} O_n = M \). Define

\[
    d(x) = \lim_{n \to \infty} d(x, M \setminus O_n), \quad x \in M,
\]

where \( d(x, A) \) is the Riemannian distance from \( x \) to set \( A \). (a) Show that either \( d(x) = \infty \) for all \( x \in M \) or \( d(x) < \infty \) for all \( x \in M \). (b) if \( d(x) = \infty \) for all \( x \in M \), then \( M \) is complete.

**Problem 7.** State and prove Gauss’ lemma about the geodesic polar coordinates of a Riemannian manifold.