GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, JUNE 2011.
ANSWER 6 QUESTIONS.

Question 1.
Let X be a connected locally contractible topological space with a base point $x \in X$.

(1) Let S be a set with an action of $\pi_1(X, x)$.
Explain how to construct a covering space $\Phi(S) \to X$, with the property that the fibre $\Phi(S)_x$ at x is S, and the monodromy action of $\pi_1(X, x)$ on $\Phi(S)_x$ coincides with the given action on S. (You may assume the existence of a universal cover, if you need it).
(2) How does the fundamental group of $\Phi(S)$ relate to the action of $\pi_1(X, x)$ on S?
(3) Suppose that $X = S^1 \vee S^1$. The fundamental group of X is the free group on two generators α, β. Consider the action of $\pi_1(X, x)$ on the set $S = \{1, 2, 3\}$ under which α acts by the cycle $(1\,2\,3)$ and β acts by the transposition $(1\,2)$. Describe explicitly (e.g. draw a picture) the covering space $\Phi(S)$ and the map $\Phi(S) \to X$.

Question 2. (1) State van Kampen's theorem.
(2) Use van Kampen's theorem to calculate the fundamental group of the surface Σ obtained from the 2-sphere by removing 2 discs and gluing in 2 Möbius bands.

Question 3.
Let $SL_2(\mathbb{R})$ denote the Lie group of 2×2 real matrices with determinant one.

Consider the action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2 = \mathbb{C}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

(here we are identifying $\mathbb{R}^2$ with $\mathbb{C}$).

Every action of a Lie group G on a manifold M leads to a Lie algebra homomorphism $\text{Lie}(G) \to \text{Vect}(M)$. Describe explicitly the Lie algebra homomorphism

$$\mathfrak{sl}_2(\mathbb{R}) \to \text{Vect}(\mathbb{R}^2)$$

corresponding to this action.

Question 4.
Let M be a smooth manifold, and let $E \to M$ be a smooth vector bundle.

(1) Define the notion of a *connection* on a vector bundle E.
(2) Define the *torsion* of a connection on the tangent bundle to M.
(3) Suppose that M is equipped with a Riemannian metric. What properties characterize the Levi-Civita connection on M?
(4) Consider the Riemannian manifold
$$M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$
with metric
$$g = y^{-2}\left(\mathrm{d}x^{\otimes 2} + \mathrm{d}y^{\otimes 2}\right).$$
Calculate the Levi-Civita connection on M.

Question 5.
Consider the connection on the trivial rank 2 vector bundle $\underline{\mathbb{R}}^2$ on $\mathbb{R}^2$, given by
$$\nabla = \nabla^{triv} + A\mathrm{d}x + B\mathrm{d}y$$
where
$$A = \begin{pmatrix} y & 1 \\ 0 & y \end{pmatrix} \quad B = \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}$$

(1) Calculate the curvature of ∇.
(2) Compute the holonomy of ∇ around the loop $\gamma(\theta) = (\sin\theta, \cos\theta)$.

Question 6. (1) Calculate the compactly supported cohomology groups of $\mathbb{CP}^2 \setminus \{p\}$, where p is any point in $\mathbb{CP}^2$.
(2) Using intersection theory, calculate the ring structure on these compactly supported cohomology groups.

Question 7.
Use the Mayer-Vietoris sequence to calculate the de Rham cohomology groups of $S^n \times S^m$, for all $n, m \geq 1$.

(You may use without proof any facts you know about the de Rham cohomology groups of spheres).