Question 1.
Let $X$ be a connected locally contractible topological space with a base point $x \in X$.

(1) Let $S$ be a set with an action of $\pi_1(X, x)$.

   Explain how to construct a covering space $\Phi(S) \to X$, with the property that the fibre $\Phi(S)_x$ at $x$ is $S$, and the monodromy action of $\pi_1(X, x)$ on $\Phi(S)_x$ coincides with the given action on $S$. (You may assume the existence of a universal cover, if you need it).

(2) How does the fundamental group of $\Phi(S)$ relate to the action of $\pi_1(X, x)$ on $S$?

(3) Suppose that $X = S^1 \lor S^1$. The fundamental group of $X$ is the free group on two generators, $\alpha, \beta$. Consider the action of $\pi_1(X, x)$ on the set $S = \{1, 2, 3\}$ under which $\alpha$ acts by the cycle $(1 \ 2 \ 3)$ and $\beta$ acts by the transposition $(1 \ 2)$. Describe explicitly (e.g. draw a picture) the covering space $\Phi(S)$ and the map $\Phi(S) \to X$.

Question 2.  
(1) State van Kampen’s theorem.

(2) Use van Kampen’s theorem to calculate the fundamental group of the surface $\Sigma$ obtained from the 2-sphere by removing 2 discs and gluing in 2 Möbius bands.

Question 3.
Let $SL_2(\mathbb{R})$ denote the Lie group of $2 \times 2$ real matrices with determinant one.

Consider the action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2 = \mathbb{C}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

(here we are identifying $\mathbb{R}^2$ with $\mathbb{C}$).

Every action of a Lie group $G$ on a manifold $M$ leads to a Lie algebra homomorphism $\text{Lie}(G) \to \text{Vect}(M)$. Describe explicitly the Lie algebra homomorphism

$$\mathfrak{sl}_2(\mathbb{R}) \to \text{Vect}(\mathbb{R}^2)$$

corresponding to this action.
Question 4.
Let $M$ be a smooth manifold, and let $E \to M$ be a smooth vector bundle.

1. Define the notion of a connection on a vector bundle $E$.
2. Define the torsion of a connection on the tangent bundle to $M$.
3. Suppose that $M$ is equipped with a Riemannian metric. What properties characterize the Levi-Civita connection on $M$?
4. Consider the Riemannian manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with metric $g = y^{-2} \left( dx^{\otimes 2} + dy^{\otimes 2} \right)$.
Calculate the Levi-Civita connection on $M$.

Question 5.
Consider the connection on the trivial rank 2 vector bundle $\mathbb{R}^2$ on $\mathbb{R}^2$, given by
$$\nabla = \nabla^{\text{triv}} + A dx + B dy$$
where
$$A = \begin{pmatrix} y & 1 \\ 0 & y \end{pmatrix}, \quad B = \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}$$

1. Calculate the curvature of $\nabla$.
2. Compute the holonomy of $\nabla$ around the loop $\gamma(\theta) = (\sin\theta, \cos\theta)$.

Question 6. 
1. Calculate the compactly supported cohomology groups of $\mathbb{C}P^2 \setminus \{p\}$, where $p$ is any point in $\mathbb{C}P^2$.
2. Using intersection theory, calculate the ring structure on these compactly supported cohomology groups.

Question 7.
Use the Mayer-Vietoris sequence to calculate the de Rham cohomology groups of $S^n \times S^m$, for all $n, m \geq 1$.

(You may use without proof any facts you know about the de Rham cohomology groups of spheres).