Geometry Topology Preliminary Examination

June 2012

This examination has eight problems on two pages. Do all the problems.

1. a) Classify two-dimensional real vector bundles on the two-sphere S^2 up to isomorphism.

b) Identify the tangent bundle TS^2 within your classification.

c) Is it possible to construct a two-dimensional real vector bundle on S^2 such that a generic section has one transverse intersection point with the zero section?

2. a) Let X be the connect sum of two copies of the real projective plane $\mathbb{R}\mathrm{P}^2$. Calculate $\pi_1(X)$.

b) Find a cover $Y \to X$ with fundamental group the integers $\mathbb{Z}$. Show that Y is homotopy equivalent to the circle S^1.

3. a) Construct a homeomorphism

$$GL_n(\mathbb{R}) \simeq O_n(\mathbb{R}) \times GL_n(\mathbb{R})/O_n(\mathbb{R})$$

.

b) Identify $GL_n(\mathbb{R})/O_n(\mathbb{R})$ with the space $Q_n(\mathbb{R})$ of positive definite quadratic forms on $\mathbb{R}^n$. Show that $Q_n(\mathbb{R})$ is convex and hence contractible

4. a) Give a definition of a connection on a manifold. Define what it means for a connection to be symmetric and to be compatible with a Riemannian metric.

c) Show that a Riemannian manifold has a unique connection that is symmetric and compatible with the metric.

5. Let ω be a symplectic form on a manifold M, i.e. a closed non-degenerate 2-form. Let $H : M \to \mathbb{R}$ be a smooth function and let X be the Hamiltionian vector field associated to H, so that $dH = i_X\omega$. Show that H and ω are invariant under the flow defined by X.

There are more problems overleaf

6. Let V be a Killing field on a Riemannian manifold M. This means that there is a one parameter group of isometries ψ_s such that

$$V(p) = \frac{d}{ds}\psi_s(p)\bigg|_{s=0}$$

for each $p \in M$. Show that if γ is a geodesic in M, then $\langle \dot{\gamma}(t), V(\gamma(t)) \rangle$ is independent of t.

7. Let Σ be a compact oriented surface and let $\Gamma \subseteq \Sigma$ a subspace homeomorphic to a one point union of circles.

a) Assume that there is a nontrivial loop in Γ whose homology class is trivial in $H_1(\Sigma)$. Show that Σ/Γ is not homotopy equivalent to a compact manifold.

b) Find an example when Σ/Γ is homotopy equivalent to a compact manifold.

8. For a given integer $d > 0$, compute the homology of

$$X = \{(z_0, z_1, w) \in \mathbb{C}^3 | z_0 \neq 0 \,\text{or}\, z_1 \neq 0; w \neq 0\}/\sim$$

where

$$(z_0, z_1, w) \sim (\lambda z_0, \lambda z_1, \lambda^d w)$$

for any $\lambda \neq 0$ in $\mathbb{C}$.