GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, SEPTEMBER 2011.

Question 1.
Let $\text{Gr}(2, 4)$ denote the Grassmannian of two-dimensional planes in $\mathbb{R}^4$. Let $\text{GL}_4(\mathbb{R})$ be the general linear group of invertible linear transformations of $\mathbb{R}^4$. Let $P = \langle e_1, e_2 \rangle \in \text{Gr}(2, 4)$ be the two-dimensional plane spanned by the basis vectors $e_1, e_2 \in \mathbb{R}^4$.

(1) Find an open neighborhood $U \subset \text{Gr}(2, 4)$ containing $P$, and which is homeomorphic to an open subset of some $\mathbb{R}^n$.
(2) Show $\text{GL}_4(\mathbb{R})$ acts transitively on $\text{Gr}(2, 4)$. What is the stabilizer of $P$?
(3) Show $\text{Gr}(2, 4)$ is a smooth manifold.

Let $X \subset \text{Gr}(2, 4)$ denote the subspace of planes $Q \in \text{Gr}(2, 4)$ such that $\dim(Q \cap P) \geq 1$.

(4) Show that the complement $\text{Gr}(2, 4) \setminus X$ is contractible.
(5) Find equations for the intersection of $X$ with your open neighborhood $U \subset \text{Gr}(2, 4)$ from part 1.
(6) Show $X$ is not a smooth manifold at $P$.

Question 2.
Consider the two-dimensional sphere $S^2 = \mathbb{C} \cup \{\infty\}$.
Let $0 \in \mathbb{C} \subset S^2$ denote zero.

Let $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{C}^\times$ be the cyclic subgroup of $n$th roots of unity.

Let $X_n = S^2 \setminus \mathbb{Z}/n\mathbb{Z}$ denote the complement of the $n$th roots of unity.

(1) For $n = 1, 2, 3, \ldots$, calculate $\pi_1(X_n, 0)$ in terms of generators and relations.
(2) For $n = 2$, why is there a canonical isomorphism $\pi_1(X_2, 0) \simeq \pi_1(X_2, x)$ for any $x \in X_2$?
(3) For $n = 2$, observe that $\mathbb{Z}/3\mathbb{Z} \subset \mathbb{C}^\times$ acts on $X_3$ by multiplication. Calculate the induced action on $\pi_1(X_3, 0)$ in terms of generators and relations.
(4) For $n = 4$, classify all (not necessarily connected) two-fold covers of $X_4$.

Question 3.
Let $X = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle x, y \rangle = 0, \|x\| = 1, \|y\| = 1\} \subset S^2 \times S^2$
(where $\langle x, y \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^3$).

Use the Mayer-Vietoris sequence to compute the cohomology of $X$.

You may use any results you know about the cohomology of spheres and tori, as long as they are stated precisely.
Question 4.
Let $M$ be a smooth compact oriented manifold, and let $f : M \to M$ be a diffeomorphism. A point $p \in M$ is a fixed point of $f$ if $f(p) = p$. We say that $f$ is regular if the derivative $D_p f$ of $f$ at $p$, which is a linear automorphism of $T_p M$, has no fixed points.

For each fixed point $p$, define a number $L(p)$ to be 1 if $\det D_p f > 0$, and $-1$ if $\det D_p f < 0$.

Let $\Gamma_f \subset M \times M$ be the graph of $f$, which is the image of the embedding

$$\text{Id} \times f : M \to M \times M$$

$$x \mapsto (x, f(x)).$$

Note that $\Gamma_f$ is naturally oriented, because $M$ is.

Let $\triangle \subset M \times M$ be the diagonal, that is, the graph of the identity map.

(1) Show that the set of fixed points of $f$ is the intersection of $\Gamma_f$ with $\triangle$.

(2) Show that $f$ is regular if and only if $\Gamma_f$ intersects $\triangle$ transversely.

(3) Let $[\Gamma_f]$ and $[\triangle]$ denote the fundamental classes of these submanifolds of $M \times M$. Supposing that $f$ is regular, use intersection theory to prove that

$$\sum_{p \in \text{Fix}(f)} L(p) = \int_{M \times M} [\Gamma_f] \wedge [\triangle].$$

(4) Deduce that the Lefschetz number $L(f) = \sum_{p \in \text{Fix}(f)} L(p)$ of a regular diffeomorphism only depends on the smooth homotopy class of $f$.

(5) Prove that every diffeomorphism $f$ of $S^2$ which is homotopic to the identity has at least one fixed point.

Question 5.
Define a submanifold $H \subset \mathbb{R}^9$ as follows: $H_3 \subset GL(3, \mathbb{R}) \subset Mat(3 \times 3, \mathbb{R}) \cong \mathbb{R}^9$ is given by the upper-triangular $3 \times 3$ matrices with 1’s along the diagonal. Note $H \cong \mathbb{R}^3$, coordinatized by writing $h \in H$ as

$$h = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Note, too, that $H$ is a closed subgroup of a Lie group, therefore a Lie group itself.

A basis for the tangent space $T_e H$ to $H$ at the identity at the identity $e = (x = 0, y = 0, z = 0)$ is given by the three vectors $\partial_x|_e, \partial_y|_e, \partial_z|_e$.

Let $U_e$ and $V_e$ denote elements of $T_e H$; we can write $U_e$ and $V_e$ as

$$U_e = a \partial_x|_e + b \partial_y|_e + c \partial_z|_e$$

$$V_e = a' \partial_x|_e + b' \partial_y|_e + c' \partial_z|_e$$

for some $a, a', b, b', c, c' \in \mathbb{R}$. 

• Use the definition of left-invariance and the Lie group structure to write down
the left-invariant vector fields $U$ and $V$ corresponding to $U_e$ and $V_e$.

In this way, you have constructed a map $\Psi : T_e H \rightarrow \text{Lie}(H)$, where $\text{Lie}(H)$ is the set of
left-invariant vector fields.

• We can think of the vectors $U_e, V_e \in T_e H$ as strictly upper triangular $3 \times 3$
matrices, in an evident way. Thus, we can define the matrix product $U_e \cdot V_e$
and $V_e \cdot U_e$. Verify that

$$\Psi(U_e \cdot V_e - V_e \cdot U_e) = [U, V],$$
i.e. that $\Psi$ intertwines the commutator bracket with the Lie bracket.

**Question 6.**

Let $M$ be a manifold and let $\pi : T^*M \rightarrow M$ be the cotangent bundle with fibration
map $\pi$. Define the *canonical one-form* on $T^*M$ (a one-form on a bundle which is itself a
space of one-forms) by the formula

$$\Theta_\xi(v) = \xi(\pi_*v).$$

Here $\xi$ is a point of $T^*M$ and $v \in T_\xi(T^*M)$ is a tangent vector at $\xi$. Now let $M = \mathbb{R}^2$ and let $(x, y, \xi_1, \xi_2)$ coordinatize the cotangent bundle $T^*\mathbb{R}^2 \cong \mathbb{R}^4$ consisting of
covectors $\xi_{(x,y)} = \xi_1 dx|_{(x,y)} + \xi_2 dy|_{(x,y)}$.

• Verify that in these coordinates, the one-form $\Theta$ defined above is $\Theta = \xi_1 dx + \xi_2 dy$.

Now switch to polar coordinates $(r, \theta)$ in the fibers, $\xi_1 = r \cos \theta, \xi_2 = r \sin \theta$, and
define the unit tangent bundle $S^* = \{ r = 1 \}$, with $i : S^* \hookrightarrow T^*\mathbb{R}^2$ the inclusion map.

• Using the coordinates $(x, y, \theta)$, write down $i^*\Theta$, i.e. the restriction of $\Theta$ to $S^*$.

Let us define the one-form $\alpha = i^*\Theta$ on $S^*$ to be the form you wrote down above. With
these definitions, a curve $C \hookrightarrow S^*$ in $S^*$ is said to be *Legendrian* if the restriction of
$\alpha$ to the curve is zero. Now let us use the Euclidean metric on $\mathbb{R}^2$ so that we may identify
the unit tangent bundle $S \subset T\mathbb{R}^2$ and the unit cotangent bundle $S^*$, and we will
continue to use the coordinates $(x, y, \theta)$. A parametrized curve $(x(t), y(t))$ in $\mathbb{R}^2$
induces a curve in $S \cong S^*$ by setting $\theta(t) = \tan^{-1}(y'(t)/x'(t)) - \pi/2) = - \cot^{-1}(y'/x')$
to be the normal direction (effected here by the shift by $\pi/2$).

• Verify that the parametrized cusp

$$x(t) = t^2, \quad y(t) = t^3, \quad \theta(t) = - \cot^{-1}(y'/x') \quad t \in \mathbb{R}$$
in $\mathbb{R}^2$ induces a Legendrian curve in $S$ for all $t$. 