Do at least two problems from each group, and more if you can.

**Group I**

1) Give an example of a covering space $X \to Y$ where $Y$ is the wedge of three circles and $\pi_1(X)$ is the dihedral group $a^2 = 1, b^4 = 1, aba = b^3$.

2) Fix a natural number $N > 0$. Let

\[ X = \{(u, \zeta) \mid u \in U(2, \mathbb{C}), \zeta \in \mathbb{C}, \det(u) = \zeta^N\}. \]

Let $p$ be the projection $(u, \zeta) \mapsto u$. Show that there is no continuous map $q : U(2, \mathbb{C}) \to X$ such that $pq = \text{id}$.

3) Let $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and let

\[ X = (D \times S^1) - (L_1 \cup L_2 \cup L_3 \cup L_4), \]

where

\[ L_1 = \left\{(z_1, z_2) \left| z_1 = -\frac{1}{2}\right. \right\}, \quad L_2 = \left\{(z_1, z_2) \left| z_1 = \frac{1}{2}\right. \right\}, \]

\[ L_3 = \left\{(z_1, z_2) \left| z_1 - \frac{1}{2} = \frac{1}{2}, z_2 = 1\right. \right\}, \quad L_4 = \left\{(z_1, z_2) \left| z_1 + \frac{1}{2} = \frac{1}{2}, z_2 = 1\right. \right\}. \]

Compute $\pi_1(X)$.

**Group II**

1) Let $G$ be a (finite-dimensional, not necessarily connected) Lie group, with identity element $e \in G$. Let $m : G \times G \to G$ be the group multiplication map.

   (a) Via the usual identification $T_{(e,e)}(G \times G) \cong T_eG \oplus T_eG$, show that $dm_e : T_eG \oplus T_eG \to T_eG$ is given by

   \[ dm_e(X,Y) = X + Y, \]

   for every $X, Y \in T_eG$.

   Let now $i : G \to G$ be the inversion map of $G$.

   (b) Show that for every $X \in T_eG$ we have

   \[ di_e(X) = -X. \]

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth positive function, and consider the surface of revolution

\[ M = \{(f(u) \cos v, f(u) \sin v, u) \in \mathbb{R}^3 \mid u \in \mathbb{R}, 0 \leq v < 2\pi\}. \]

   (a) Show that $M$ is a submanifold of $\mathbb{R}^3$. 

(b) Let \( \iota : M \hookrightarrow \mathbb{R}^3 \) be the inclusion. Using \((u, v)\) as global coordinates on \( M \), write down the metric \( g = \iota^* g_{\text{Eucl}} \) induced from the Euclidean metric on \( \mathbb{R}^3 \).

(c) Write down the same metric explicitly when \( f(x) = e^x \).

3) Consider the vector fields

\[
X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},
\]

in \( \mathbb{R}^3 \) with the standard coordinates \((x, y, z)\).

(a) Find local coordinates \((u, v, w)\) in a neighborhood of \((x, y, z) = (1, 0, 0)\), such that in these coordinates we have

\[
X = \frac{\partial}{\partial u}, \quad Y = \frac{\partial}{\partial v}.
\]

(b) Is it possible to do the same for the vector fields

\[
X' = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad Y' = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.
\]

**Group III**

1) Compute \( H^k(\mathbb{P}^8; \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \), for all \( k \).

2) Show that if \( \pi : \mathbb{C}P^{2n} \to X \) is a covering space, then \( X = \mathbb{C}P^{2n} \) and \( \pi \) is the identity.

3) Let \( M \) be a compact orientable manifold of dimension \( n \geq 2 \), and \( p \in M \). Suppose you know the de Rham cohomology groups of \( M \), determine those of \( M \setminus \{p\} \).