1. **Geometry**

**Answer 3 of the following questions.**

1. (a) Define the *torsion* of a connection on the tangent bundle of a manifold $M$.

(b) Prove that there exists a unique torsion-free connection on the tangent bundle of a Riemannian manifold $M$ which is compatible with the metric.

(c) If $\partial_i, \partial_j$ is a coordinate frame of vector fields, give a formula for $\nabla_{\partial_i} \partial_j$ in terms of the components $g_{ij} = g(\partial_i, \partial_j)$ of the metric.

2. Let $G$ be a compact Lie group.

(a) Show that $G$ has a bi-invariant Riemannian metric $\sigma$.

(b) Show that the integral curves of left-invariant vector fields on $G$ are geodesics for $\sigma$.

(c) Show that if $Z$ is a left-invariant vector field on $G$, then $\nabla_Z Z = 0$, where $\nabla$ is the Levi-Civita connection for $\sigma$.

(d) Show that if $X$ and $Y$ are left-invariant vector fields on $G$,

$$\nabla_X Y = \frac{1}{2} [X, Y].$$

3. (a) State Cartan’s formula for the Lie derivative of a differential form.

(b) Let $Y$ be an integrable vector field on $M$, and let $\phi_t : M \to M$ be the corresponding one-parameter family of diffeomorphisms. How does the operator $\phi_t^* : \Omega^k(M) \to \Omega^k(M)$ relate to the Lie derivative by $Y$?

(c) Suppose that $\omega$ is a closed two-form on a manifold $M$ and $f : M \to \mathbb{R}$ is a differentiable function. If there is an integrable vector field $Y_f$ satisfying $df(X) = \omega(Y_f, X)$ for all vector fields $X$, show that its flow preserves $\omega$.

4. Consider the distribution $\mathcal{D}$ defined by the following two vector fields on $\mathbb{R}^3$ with coordinates $x, y, z$:

$$V_1 = \partial_y + z\partial_x$$

$$V_2 = \partial_z + y\partial_x$$

(a) Express the distribution $\mathcal{D}$ as the kernel of a closed one-form.

(b) Show that the distribution is integrable.

(c) Conclude (how?) that we can find an integral manifold $M_p$ through every point $p \in \mathbb{R}^3$, and find $M_p$ for $p = (1, 0, 0)$. 


2. Topology

Answer 3 of the following questions.

(5) Let $X$ be a topological space which can be written as

$$X = U_1 \cup U_2 \cup U_3$$

with each $U_i$ open in $X$. Suppose $U_i$ and $U_i \cap U_j$ are contractible for $1 \leq i, j \leq 3$. Show

$$\tilde{H}_n X \cong \tilde{H}_{n-2}(U_1 \cap U_2 \cap U_3).$$

(6) Let $p : \tilde{X} \to X$ be the universal cover of a path-connected and locally-path connected space $X$ and let $A \subseteq X$ be a path-connected and locally path-connected subspace. Let $\tilde{A}$ be a path component of $p^{-1}(A)$. Show that $\tilde{A} \to A$ is a covering space and that the image of

$$\pi_1(\tilde{A}, b) \to \pi_1(A, a)$$

is the kernel of $i_* : \pi_1(A, a) \to \pi_1(X, a)$. Here $b \in \tilde{A}$ is any basepoint and $a = p(b)$.

(7) Let $X$ be the topological space obtained as the quotient of the sphere $S^2$ under the equivalence relation $x \sim -x$ for $x$ in the equatorial circle.

(a) Describe a CW complex whose underlying space is $X$.
(b) Write down the CW chain complex of $X$.

(8) Let $X$ be the space obtained from a torus $T = S^1 \times S^1$ be attaching a Möbius band $M$ by a homeomorphism from the boundary circle of $M$ to $S^1 \times \{x_0\} \subseteq T$. Compute $\pi_1(X, x_0)$. 