Do all of the following questions.

1. Let $X$ be a path-connected space with abelian fundamental group $A$ and contractible universal cover. Suppose $f : Y \rightarrow X$ is a continuous map so that

$$0 = f^* : H^1(X, A) \longrightarrow H^1(Y, A).$$

Prove $f$ is null-homotopic.

2.i.) Let $X$ be a CW complex and suppose that for all non-negative integers $n$, $X$ has only finitely many $n$-cells. Prove that if $\tilde{H}^*_p(X, \mathbb{F}_p) = 0$ for all primes $p$, then $\tilde{H}^*_p(X, \mathbb{Q}) = 0$.

ii.) Show by example that the hypothesis on the number of cells in $X$ is necessary.

3.i.) Let $M$ be a compact, connected manifold without boundary of even dimension $2n$. Show that there is a unique class $v \in H^n(M, \mathbb{F}_2)$ so that

$$v \sim u = u^2$$

for all $u \in H^n(M, \mathbb{F}_2)$.

ii.) Find $v$ if $M = \mathbb{R}P^2 \times \mathbb{R}P^2$.

4. One form of the Poincaré conjecture states that if a smooth 3-manifold $M$ is homotopy equivalent to $S^3$, then $M$ is diffeomorphic to $S^3$.

i.) Assuming this version of the Poincaré conjecture, show that the universal cover of any connected 3-manifold $M$ is either $S^3$ or contractible.

ii.) Now assume $M$ is orientable. Show that $M$ is parallelizable; that is, there exists a trivialization of the tangent bundle, $TM \cong M \times \mathbb{R}^3$. 

Over
5. Let $X$ and $Y$ be two spaces and define

$$X \ast Y = X \times Y \times [-1, 1]/\sim$$

where $\sim$ is the smallest equivalence relation so that

$$(x, y, -1) \sim (x', y, -1) \quad \text{and} \quad (x, y, 1) \sim (x, y', 1).$$

Show that there is a short exact sequence

$$0 \to \tilde{H}_{n+1}(X \ast Y) \to \tilde{H}_n(X \times Y) \xrightarrow{p} \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \to 0$$

with $p(a) = (p^1(a), p^2(a))$. Here $p^i$ are the two projections from the product.

6. Let $X = S_1 \cup S_2 \subseteq \mathbb{R}^3$ be the union of two spheres of radius 2, one about $(1,0,0)$ and the other about $(-1,0,0)$. Thus

$$S_1 = \{(x, y, z) \mid (x - 1)^2 + y^2 + z^2 = 4\}$$

and

$$S_2 = \{(x, y, z) \mid (x + 1)^2 + y^2 + z^2 = 4\}.$$

i.) Give a description of $X$ as a CW complex.

ii.) Write out the cellular chain complex of $X$.

iii.) Calculate $H_*(X, \mathbb{Z})$.

7. Consider the tautological $\mathbb{H}$-line bundle on quaternionic projective space, $E \to \mathbb{HP}^n$. For $n = 1$, $E$ thus defines a 4-dimensional real vector bundle over $\mathbb{HP}^1 \simeq S^4$. Compute the value of the pairing $\langle p_1(E), [S^4] \rangle$. Show whether or not $E$ is equivalent to the tangent bundle of $S^4$. 