Solve two problems from each group. If you do more than this, your best two scores of each group will count.

All manifolds are assumed to be connected.

Group I

1) Write down an explicit embedding of the fundamental group of the sphere with three handles to the fundamental group of the sphere with two handles as a normal subgroup with quotient $\mathbb{Z}/2\mathbb{Z}$.

2) Let $X$ be the topological space obtained from $\mathbb{R}^3$ by deleting a circle together with a chord. Prove that $X$ is homotopy equivalent to $S^2 \vee S^1 \vee S^1$.

3) Construct a covering space $Y \to S^1 \vee S^1 \vee S^1 \vee S^1$ with the group of deck transformations equal to $(\mathbb{Z}/2\mathbb{Z})^2$. Compute $\pi_1(Y, y_0)$ where $y_0$ is a pre-image of the base point of $S^1 \vee S^1 \vee S^1 \vee S^1$.

Group II

1) Let $D$ be the distribution on $\mathbb{R}^3$ spanned by the vector fields

$$X = ye^x \frac{\partial}{\partial y} - \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial x}.$$  

(a) Write down an integral submanifold for $D$ passing through $(1,0,0)$

(b) Is $D$ integrable?

(c) Can you write down an integral submanifold for $D$ passing through $(1,1,1)$?

2) Let $M$ be a connected smooth manifold. Show that the action of the diffeomorphism group of $M$ on $M$ is transitive, i.e. show that given any two points $p, q \in M$ there is a diffeomorphism $F : M \to M$ with $F(p) = q$.

3) Let $S^n$ be the $n$-dimensional sphere.

(a) Show that there is no nonvanishing smooth vector field on $S^n$ for $n \geq 2$ even.

(b) Show that there is a nonvanishing smooth vector field on $S^n$ for $n \geq 1$ odd.

(there are more problems overleaf)
Group III

1) Let $A \subset X$ be a subspace of a topological space. Show that $H^1(X, A; \mathbb{Z})$ is torsion-free.

2) Let $M$ be a compact orientable manifold of dimension $2k$, $k \in \mathbb{N}$.
   (a) Show that the parity of the Euler characteristic $\chi(M)$ is equal to the parity of $\dim \mathbb{R} H^k(M; \mathbb{R})$.
   (b) Assume now that $k$ is odd, and show that $\dim \mathbb{R} H^k(M; \mathbb{R})$ (and hence $\chi(M)$) is even.
   (c) Find a compact orientable 4-manifold $M$ such that $\chi(M)$ is odd.

3) Let $M_1, M_2$ be two compact $n$-manifolds and for each $j = 1, 2$ let $B_j \subset M_j$ be open subsets with $\psi_j : B_j \to B_1(0) \subset \mathbb{R}^n$ diffeomorphisms. Let $B'_j = \psi_j^{-1}(\overline{B_{1/2}(0)})$ and $M'_j = M_j \setminus B'_j$. Let the connected sum of $M_1$ and $M_2$ be
   \[ M_1 \# M_2 = (M'_1 \sqcup M'_2) / \sim, \]
   where the equivalence relation $\sim$ identifies $x_1 \in B_1 \setminus B'_1$ with $x_2 \in B_2 \setminus B'_2$ iff $\psi_1(x_1) = \psi_2(x_2)$. We have that $M_1 \# M_2$ is also a compact $n$-manifold. Show that the Euler characteristic of $M_1 \# M_2$ is given by
   \[ \chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n). \]