Do all six problems.

1. Prove any continuous map $f : \mathbb{RP}^2 \to S^1 \times S^3$ is homotopic to a constant map.

2. a) Let $M$ be a connected, compact, $n$-manifold without boundary, $n \geq 2$. Suppose that $M$ cannot be oriented. Show that $H_{n-1}(X, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

b) Show any simply-connected compact $n$-manifold without boundary, $n \geq 2$, is orientable.

3. Let $T_n$ be an $n$-holed torus with a chosen orientation. Are the following statements true or false? (If true, supply an example, if false, give an argument.)

a) There exists a degree 1 map $T_1 \to T_2$.

b) There exists a degree 1 map $T_3 \to T_2$.

Recall that a “degree 1 map” takes the orientation class of the source to the orientation class in the target.

4. Let $\Sigma \subseteq \mathbb{R}^2$ be a subspace homeomorphic to $S^1$. Then, by the Jordan Curve Theorem, $\mathbb{R}^2 - \Sigma$ is the disjoint union of subspaces $U$ and $B$ with $U$ unbounded and $B$ bounded. Furthermore $B \cup \Sigma$ is homeomorphic to the disk $D^2$.

Let $x \in \mathbb{R}^2 - \Sigma$ and

$$i_*: H_1(\Sigma) \to H_1(\mathbb{R}^2 \setminus \{x\})$$

be the homomorphism induced by inclusion. Prove:

a) If $x \in U$, then $i_* = 0$.

b) If $x \in B$, then $i_*$ is an isomorphism.

5. Let $N$ be a compact manifold without boundary of dimension $n \geq 1$. Let $x_0 \in N$ be a fixed element. Show that the two maps $i_1, i_2 : N \to N \times N$ given by

$$i_1(y) = (x_0, y) \quad \text{and} \quad i_2(y) = (y, x_0)$$

are not homotopic.

6. Let $N = \mathbb{CP}^2 - D$ where $D$ is an open disk with the property that the boundary of $N$ is diffeomorphic to $S^3$. Define a new manifold $M = N \cup_{S^3} N$ where we have identified the two boundary $S^3$’s via an orientation-reversing diffeomorphism.

a) What is the integral cohomology ring $H^*(M, \mathbb{Z})$?

b) Would your answer be different if we used an orientation preserving diffeomorphism?