Part A: Do TWO of the following 3 problems. (15 points each.)

1. Let \( X \) be a compact embedded \( C^\infty \) submanifold of \( \mathbb{R}^n \) with codimension at least one. Let \( NX \) be the normal bundle of \( X \) consisting of all pairs \( (x,v) \in X \times \mathbb{R}^n \) such that \( v \) is perpendicular to all tangent vectors to \( X \) at \( x \). Define \( f : NX \to \mathbb{R}^n \) by \( f(x,v) = x + v \). Show that the derivative of \( f \) at \( (x,0) \) is a linear isomorphism for each \( x \in X \). Then show that there is an \( \varepsilon > 0 \) such that the restriction of \( f \) to \( \{(x,v) \in NX : \|v\| < \varepsilon\} \) is a diffeomorphism onto its image. (You may use the fact that \( NX \) is a \( C^\infty \) manifold of dimension \( n \).)

2. Suppose that \( X \) and \( Y \) are \( C^\infty \) manifolds, \( X \) is compact, and \( f : X \to Y \) is a \( C^\infty \) map that is transversal to a compact \( C^\infty \) submanifold \( Z \) of \( Y \). Show that if \( F : X \times [0,1] \to Y \) is a \( C^\infty \) map with \( F(x,0) = f(x) \) for all \( x \in X \), then the map \( f_s \) defined by \( f_s(x) = F(x,s) \) is transversal to \( Z \) for all small enough \( s > 0 \).

3. Suppose \( X \) is a compact \( C^\infty \) manifold, \( Y \) is a connected \( C^\infty \) manifold with the same dimension as \( X \), and \( f : X \to Y \) is a \( C^\infty \) map whose mod 2 degree is nonzero. Prove that \( f \) is onto.

Part B: Do TWO of the following 3 problems. (15 points each.)

4. Define the cellular chain complex of a CW complex (in particular, define the boundary map of this complex). Provide the 2-torus \( S^1 \times S^1 \) with a CW structure and exhibit the associated cellular chain complex. Compute the homology and cohomology of this chain complex (and thus of \( S^1 \times S^1 \)).

5. Let \( X \) be a finite simplicial complex. Give several different (but equivalent) definitions of the Euler characteristic and verify that each gives the same number for \( X \).

6. Give the general construction of the covering space of a connected, pointed, locally path connected space \( X,x \) associated to a subgroup of \( \pi_1(X,x) \). Make this more specific by repeating this construction for the explicit example of \( X = S^1 \lor S^1 \) and the subgroup \( \mathbb{Z} \cdot s \subset \mathbb{F}(s,t) = \pi_1(S^1 \lor S^1) \), where \( \mathbb{F}(s,t) \) denotes the free group on generators \( s,t \). Reinterpret “geometrically” (i.e., in terms of covering spaces) the fact that this subgroup is not normal.

Part C: Do EACH of the following 3 problems. (10 points each.)

7. Give examples of each of the following and briefly justify each example:
   (a) Two spaces which are homotopy equivalent but not homeomorphic.
   (b) Two connected pointed spaces \( X,Y \) with \( H_1(X,\mathbb{Z}) \cong H_1(Y,\mathbb{Z}) \), \( \pi_1(X,x) \not\cong \pi_1(Y,y) \).
   (c) A CW complex \( X \) such that \( H_1(X,\mathbb{Z}) \) is an uncountable free abelian group.
   (d) A topological space which is not a CW complex.

8. Use Mayer-Vietoris to compute the integral homology of the following spaces:
   (a) The subspace of \( \mathbb{R}^3 \) given as the union of the unit sphere \( S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1 \} \) and the unit 2-disk \( D^2 = \{(x,y,0) : x^2 + y^2 \leq 1 \} \).
   (b) The connected sum of a torus and a Klein bottle.

9. Prove that the singular homology of a 2-dimensional simplicial complex \( X \) vanishes in dimensions greater than 2 (i.e., \( H_i(X,\mathbb{Z}) = 0 \), \( i > 2 \)).