Topology Preliminary Exam, September 1996

Do 2 of 3 problems from Part A and 4 of 6 problems from Part B.

Be sure to indicate which problems you are submitting.

Part A

In Part A, X will be the subset of $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$.

(A1.) (a) Show that X is a smooth manifold.
(b) Describe the normal space to X (as a subset of $\mathbb{R}^6$).

(A2.) (a) Describe the tangent space to X (as a subset of $\mathbb{R}^6$).
(b) Calculate the self intersection number of X.

(A3.) Consider the function $f : X \to \mathbb{R}^4$ given by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Let Y be the image set $f(X)$.
(a) Show that Y is a smooth manifold and that the map $f$ is smooth.
(b) Describe the tangent space to Y.
(c) Calculate the mod 2 self intersection number of Y.

Part B

(B1.) Let $\tilde{X} \to X$ be a covering map. Assume $\tilde{X}$ and X are path connected and that $\pi_1(X, x_0)$ is finite. Let $\tilde{x}_0 \in \tilde{X}$ so that $p(\tilde{x}_0) = x_0$. Show that the number of points in $p^{-1}(x_0)$ is equal to $[\pi_1(X, x_0) : p_\*\pi_1(\tilde{X}, \tilde{x}_0)]$, the index of the image of $p_\*$ in the fundamental group of X.

(B2.) Let $f : S^3 \to S^3$ be a map such that $f(-x) = -f(x)$ for all $x \in S^3$. Then $f$ induces a map $g : \mathbb{R}P^3 \to \mathbb{R}P^3$.
(a) Show that $g_\* : \pi_1(\mathbb{R}P^3, *) \to \pi_1(\mathbb{R}P^3, *)$ is an isomorphism.
(b) Show that $f_\* : H_3(S^3; \mathbb{Z}) \to H_3(S^3; \mathbb{Z})$ is multiplication by an odd integer.

(B3.) (a) Describe a map $f : S^2 \times S^2 \to S^4$ such that $f_\* : H_4(S^2 \times S^2; \mathbb{Z}) \to H_4(S^2; \mathbb{Z})$ is an isomorphism.
(b) Does there exist a map $f : S^4 \to S^2 \times S^2$ such that $f_\* : H_4(S^2; \mathbb{Z}) \to H_4(S^2 \times S^2; \mathbb{Z})$ is an isomorphism? Explain your answer.

(B4.) Give an example of a map $f : S^p \to S^q$, for some p and q with $p > q$, such that $f$ is not homotopic to a constant map. Include a proof that your map $f$ is, in fact, not homotopic to a constant map.

(B5.) Let $P$ be the projective plane and let $K$ be the Klein bottle.
(a) What are $H_*(K; \mathbb{Z})$ and $H_*(K; \mathbb{Z})$?
(b) What is $H_*(K \times P; \mathbb{Z})$?

(B6.) Let $M$ be a compact connected manifold of dimension $m$ and let $N$ be a compact connected manifold of dimension $n$, both without boundary. Suppose there is a map $f : M \to N$ which is one-to-one.
(a) Show that $m \leq n$.
(b) Show that if $m = n$, then $f$ is a homeomorphism of $M$ onto $N$. 