Do 2 of 3 problems from Part A and 4 of 6 problems from Part B.

Be sure to indicate which problems you are submitting.

Part A

1. Define **chart, atlas** and **manifold**. Define $C^\infty$ **function** from one manifold to another and define the **derivative** of such a map. Give an example of a $C^\infty$ homeomorphism whose inverse fails to have a derivative at some point.

2. (a) Define **transversal intersection** of two submanifolds.
(b) Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and let $W$ be the graph of $A$, i.e. $W = \{(v, A(v)) \in \mathbb{R}^n \times \mathbb{R}^n \mid v \in \mathbb{R}^n\}$. Let $V = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Prove that $V$ is transverse to $W$ in $\mathbb{R}^n \times \mathbb{R}^n$ if and only if $A$ is an isomorphism.

3. (a) Define **homotopy** of two continuous maps.
(b) Suppose $f(t)$ is a loop in $(X, x_0)$, i.e. a continuous function $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$. Let $g : [0, 1] \to [0, 1]$ be a continuous function satisfying $g(0) = 0$ and $g(1) = 1$. Prove that $f(g(t))$ is another loop in $(X, x_0)$, and that it is homotopic to the loop $f(t)$ relative to the base point $x_0$.

Part B

4. Prove that the boundary of the Möbius Band is a not a retract of the Möbius Band.

5. Let $X$ be the quotient space obtained by identifying the three vertices of a 2-simplex to a single point. Find the homology groups $H_k(X; \mathbb{Z})$ for all $k$.

6. For each of the following two statements, tell whether the statement is True or False. In each case, support your answer.
(a) There exists a map $f : S^2 \times S^2 \to S^4$ such that $f_* : H_4(S^2 \times S^2; \mathbb{Z}) \to H_4(S^2; \mathbb{Z})$ is an isomorphism.
(b) There exists a map $g : S^4 \to S^2 \times S^2$ such that $g_* : H_4(S^2; \mathbb{Z}) \to H_4(S^2 \times S^2; \mathbb{Z})$ is an isomorphism.

7. Show that $\mathbb{RP}^4 \times S^4$ and $\mathbb{RP}^8 \vee S^4$ do not have the same homotopy type.

8. Assume that $f : X \to Y$ is a covering map, where $X$ and $Y$ are Hausdorff spaces and $X = D^2$. Prove that $f$ is a homeomorphism.

9. Let $K$ be the Klein Bottle and let $P = \mathbb{RP}^2$, the 3-dimensional real projective space.
(a) What are $H_k(P \times K; \mathbb{Z})$, for all $k$?
(b) What are $H_k(P \times K; \mathbb{Z}/2\mathbb{Z})$, for all $k$?