Part A: Do TWO of the following 3 problems.

A1. (a) Define chart, atlas and manifold.
(b) Let $X$ denote the one point compactification of the complex numbers $C$. So $X = C \cup \{\infty\}$ and the sets $\{z : |z| > r, r > 0\} \cup \{\infty\}$ form a neighborhood basis of $\{\infty\}$. Prove that $X$ is a manifold and the function $f : X \to X$ is smooth if $f$ is defined by $f(z) = z^2$ if $z \in C$ and $f(\infty) = \infty$. In particular prove $f$ is smooth at the point $\infty$.

A2. (a) Define transversal intersection of two submanifolds.
(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are $C^\infty$ functions and that the derivatives $df_x \neq dg_x$ whenever $f(x) = g(x)$. Prove that the graphs of $f$ and $g$ are submanifolds of $\mathbb{R}^{n+1}$ and that they intersect transversally. Note: the graph of $f$ is $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$.

A3. (a) Define critical point and critical value of a smooth function. State Sard’s Theorem.
(b) Suppose $f : S^1 \to \mathbb{R}^4$ is a smooth embedding. Prove that there is a three dimensional subspace $V$ of $\mathbb{R}^4$ such that $P \circ f : S^1 \to V$ is one-to-one, where $P$ is orthogonal projection of $\mathbb{R}^4$ onto $V$.

Part B: Do EACH of the following 3 problems.

B1. Calculate $H_p(RP^3 \times RP^3; \mathbb{R})$ for all $p$ and for $\mathbb{R} = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{R} = \mathbb{Z}/3\mathbb{Z}$.

B3. A homology class $\alpha \in H_n(X; \mathbb{Z})$ is called spherical if there is a map $f : S^n \to X$ such that $f_*(\mu) = \alpha$, where $\mu$ generates $H_n(X; \mathbb{Z})$.

Which classes $\alpha \in H_{p+q}(S^p \times S^q; \mathbb{Z})$ are spherical ($p \geq 1, q \geq 1$)?

B3. Calculate $H_*(X; \mathbb{Z})$ where $X = S^2 \cup \{(0,0,t) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\} \cup (D^2 \times \{0\})$.

In words: $X$ is the union of a 2-sphere with an equatorial disk and with a line segment joining the North and South poles.

Part C: Do TWO of the following 4 problems.

C1. Can $CP^2$ be homeomorphic to a proper subspace of itself? Explain your answer.

C2. Let $p : X \to Y$ be a covering map with $X$ (and hence $Y$) path-connected and locally path-connected. Suppose there is a map $f : X \to X$ such that $p \circ f = p$. Show that $f$ is a homeomorphism.

C3. Suppose $A \subset X$ where $X$ is contractible. Suppose that $\alpha \in H^p(X,A)$ and $\beta \in H^q(X,A)$ where $p > 0$ and $q > 0$. Show that $0 = \alpha \cup \beta \in H^{p+q}(X,A)$.

C4. Prove that $S^{2n}$ cannot be a covering space of $CP^n$ if $n \geq 2$. 