Math 441/2 Preliminary Exam
September 2006

Do all of the following questions.

1. Let $M(n)$ denote the space of $n \times n$ matrices (the entries are real numbers), where n is a positive integer. Prove that the orthogonal group
$$O(n) = \{A \in M(n) \mid AA^t = I_n\}$$
is a manifold of dimension $n(n-1)/2$, where A^t is the transpose of A and I_n is the identity matrix.

2. Suppose that $f : X \longrightarrow Y$ is a smooth map between manifolds, where $\dim X \geq \dim Y$. Fix a point $c \in Y$ and let
$$V = X \times \{c\} \subset X \times Y$$
$$W = \text{graph}(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$
Prove that V and W are transveral submanifolds of $X \times Y$ if and only if c is a regular value for $f : X \longrightarrow Y$.

3. Let S^1 be the unit circle in $\mathbb{R}^2$ and let $T = S^1 \times S^1$ be the torus. Let $S^1 \subseteq T$ be the inclusion of the diagonal circle. Define a quotient space $X = T \cup_{S^1} T$ by taking two copies of the torus and identifying the two diagonal circles. Calculate the integral cohomology ring of X.

4. Let $f : \mathbb{RP}^m \to \mathbb{RP}^n$ be a continuous map with $m > n > 0$. What can you say about the induced map on fundamental groups? Prove your answer.

5. Prove that the Euler characteristic of a compact orientable manifold of dimension $4k+2$ is even. Give an example to show this statement is false in dimension 4.

6. Let M and N be compact, connected manifolds. Prove that M and N are both orientable if and only if $M \times N$ is orientable.

7. Give an example of a non-trivial line bundle on $\mathbb{RP}^n$. Be sure and prove it's not trivial.