NORTHWESTERN UNIVERSITY | Thursday, September 30, 2004

FIRST SELECTION TEST

NAME: ______________________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A1. Find all positive integers x, y such that $4^x + 5 = 9^y$.

Answer:

We will prove that the only solution is $x = y = 1$.

* *Method 1*: We have

$$9^y - 4^x = 3^{2y} - 2^{2x} = (3^y + 2^x)(3^y - 2^x) = 5\,,$$

hence $3^y + 2^x \leq 5$, and the only positive values of x, y that verify this inequality are $x = y = 1$.

* *Method 2*: We have that $4^x \equiv 0 \pmod 8$ for $x \geq 2$, and $9^y \equiv 1 \pmod 8$ for every y. Since $0 + 5 \not\equiv 1 \pmod 8$, the only posibility is $x = 1$, and $y = 1$.

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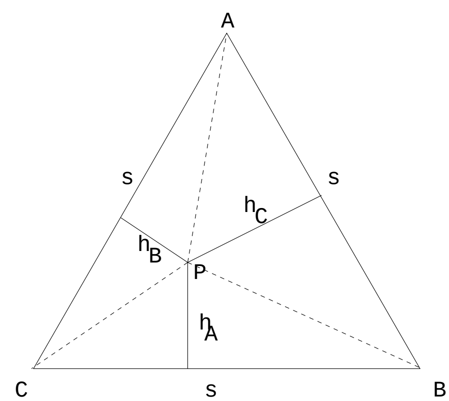
Problem A2. Prove that from any point inside an equilateral triangle, the sum of the measures of the distances to the sides of the triangle is constant.

Answer:

Let P be a point inside an equilateral triangle ABC. Let d_A, d_B, d_C the distance from P to the side opposed to A, B, C respectively, and assume $|AB| = |BC| = |CA| = s$. The area S of the triangle equals the sum of the areas of the three triangles APB, BPC and CPA respectively, i.e.

$$\begin{aligned} S &= \frac{1}{2}\,|BC|\,h_A + \frac{1}{2}\,|CA|\,h_B + \frac{1}{2}\,|AB|\,h_C \\ &= \frac{s}{2}\,(h_A + h_B + h_C)\,, \end{aligned}$$

hence $h_A + h_B + h_C = 2S/s =$ constant.



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WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A3. Let $a, b, c, d > 0$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Answer:

For $x, y > 0$ we have

$$0 \leq (x-y)^2 = (x+y)^2 - 4xy \implies \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y},$$

hence:

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{4}{a+b} + \frac{4}{c} + \frac{16}{d} \geq \frac{16}{a+b+c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

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Problem A4. Find $\lim_{n\to\infty} \prod_{k=0}^{n} \left(1 + \frac{1}{3^{2^k}}\right)$.

Answer:

If we call the product P_n we have

$$\begin{aligned}
\left(1 - \frac{1}{3}\right) P_n &= \left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3^2}\right)\dots\left(1 + \frac{1}{3^{2^n}}\right) \\
&= \left(1 - \frac{1}{3^2}\right)\left(1 + \frac{1}{3^2}\right)\dots\left(1 + \frac{1}{3^{2^n}}\right) \\
&\dots \\
&= \left(1 - \frac{1}{3^{2^{n+1}}}\right) \underset{n\to\infty}{\longrightarrow} 1\,.
\end{aligned}$$

Hence $\lim_{n\to\infty} P_n = \left(1 - \frac{1}{3}\right)^{-1} = \boxed{\frac{3}{2}}$.

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Problem A5. Prove that if a, b are two positive integers and $\sqrt{a}$ is irrational then $\sqrt{a}+\sqrt{b}$ is irrational.

Answer:

Calling $r = \sqrt{a} + \sqrt{b}$, we have $\sqrt{a} = \frac{1}{2}\left(r + \frac{a-b}{r}\right)$. So if r were rational so would be $\sqrt{a}$, contradicting the hypothesis.

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Problem A6. Prove that in the following product

$$P = (1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + x^3 + \cdots + x^{99} + x^{100})$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

Answer:

Let $p(x)$ and $q(x)$ be the following polynomials:

$$p(x) = 1 + x^2 + x^4 + \cdots + x^{98} + x^{100}\,, \quad q(x) = 1 + x^2 + x^4 + \cdots + x^{98}\,.$$

Then the given product can be written:

$$\begin{aligned} P &= [p(x) - x\,q(x)][p(x) + x\,q(x)] \\ &= [p(x)]^2 - x^2\,[q(x)]^2\,. \end{aligned}$$

That expression contains only even powers of x.