Putnam Selection Test 2008 (Answers)

1. Answer: 336. If $N = abc$, $a \le b \le c$, and $a + b = c$, we have $abc = 6(a + b + c)$, $ab(a + b) = 12(a + b)$, $ab = 12$. Possible values are $(a, b) = (1, 12), (2, 6), (3, 4)$, so $N = 1 \cdot 12 \cdot (1 + 12) = 156$, $N = 2 \cdot 6 \cdot (2 + 6) = 96$, $N = 3 \cdot 4 \cdot (3 + 4) = 84$, and the sum is $156 + 96 + 84 = 336$.

2. Answer: 348. Write the sequences $a_n = a_1 + (n-1)d$, $b_n = b_1 + (n-1)d$, so $a_nb_n = a_1b_1 + (n-1)(a_1d' + b_1d) + (n-1)^2dd'$. Hence, $\Delta(a_nb_n) = a_{n+1}b_{n+1} - a_nb_n = a_1d' + b_1d + (2n-1)dd'$ is an arithmetic sequence of difference $2dd' = (1848 - 1716) - (1716 - 1440) = -144$. Hence the sequence of differences of a_nb_n is $276, 132, -12, -156, -300, -444, -588, \ldots$, and from here we get $a_nb_n = \{1440, 1716, 1848, 1836, 1680, 1380, 936, 348, \ldots\}$.

3. Answer: 103/280. General number of derangements is $n!(1 - 1 + 1/2! - 1/3! + 1/4! - 1/5! + 1/6! - .... + (-1)^n/n!)$ by inclusion exclusion.

4. Answer: 0. The substitution $x \to 1/x$ makes $I = -I$.

5. Answer: A. Integration by parts, $\dfrac{\sin x}{x} \to \dfrac{1 - \cos x}{x^2}$. But $1 - \cos x = 2\sin^2(x/2)$.

6. Answer: 16. Every number less than 16 can be written that way. If $16 = x^2 - p$, then $p = x^2 - 16 = (x - 4)(x + 4)$, so $(x - 4) = 1$, yet then $p = 9$ is not prime.

7. Answer: 18. If $S_1 = \alpha + \beta + \gamma$, $S_2 = \alpha^2 + \beta^2 + \gamma^2$, $S_3 = \alpha^3 + \beta^3 + \gamma^3$, $P = \alpha\beta\gamma$, then by algebra $S_1^3 = S_3 + 3(S_1S_2 - S_3) + 6P$, and from here we get $6P = 7^3 - 19 - 3\cdot(7\cdot 13 - 19) = 108$, $P = 18$.

8. Answer: 296. Since $n = (n - 2) + 2$, all the n are at most 3. On the other hand, $6 = 3 + 3 = 2 + 2 + 2$, and $9 > 8$. So write $100 = 32 \cdot 3 + 2 \cdot 2$, and the answer is $32 \cdot 3^2 + 2 \cdot 2^2 = 296$.

9. Answer: $2009^{1/3} - 1$. We have $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, so $1/(x^2 + xy + y^2) = (x - y)/(x^3 - y^3)$. If $x^3 = n + 1$, and $y^3 = n$, then $1/(x^2 + xy + y^2) = x - y$. Thus the sum is $\sum_{n=1}^{2008}((n + 1)^{1/3} - n^{1/3}) = 2009^{1/3} - 1$.

10. Answer: $\{7, 1\}$. We have that $\frac{1+\sqrt{5}}{2} = \phi$ and $\frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$ are the roots of the polynomial $x^2 - x - 1$, and for $n > 1$, $f(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, and verifies the recurrence $f(n+2) = f(n+1) + f(n)$. Hence $f(n) = \{2, 3, 4, 7, 11, 18, 29, \ldots\}$ Lucas numbers. So, mod 10, we get:

$$2, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, \ldots$$

so we see that it repeats in cycles of 12. So $f(2004) = 2$, $f(2005) = 1$, $f(2006) = 3$, $f(2007) = 4$, $f(2008) = 7$, $f(2009) = 1$.

11. Answer: 98. I claim that $2y_n < x_{n+2} < y_{n+1}$ for $n > 0$. Proof by induction. For $n = 2$, we get $2 \cdot 3 < 16 < 27$. Now $x_{n+3} = 2^{x_{n+2}} < 3^{x_{n+2}} < 3^{y_{n+1}} = y_{n+2}$, and $x_{n+3} = 2^{x_{n+2}} > 2^{2y_n} = 4^{y_n} = 3^{y_n}(4/3)^{y_n} = y_{n+1}(4/3)^{y_n} > 2y_{n+1}$, if $y_n \ge 3$.