Problem A1. Prove that \((\sqrt{5} + 2)^{1/3} - (\sqrt{5} - 2)^{1/3} = 1\).

- Answer: Let \(x = (\sqrt{5} + 2)^{1/3} - (\sqrt{5} - 2)^{1/3}\). Raising to the third power, expanding and simplifying we get that \(x\) verifies the equation
\[
x^3 + 3x - 4 = 0.
\]

On the other hand we have:
\[
x^3 + 3x - 4 = (x - 1)(x^2 + x + 4).
\]
The second factor has no real roots, so the only real root of \(x^3 + 3x^2 - 4\) is 1, and the result follows.
Problem A2. Let $x$ be a real number. Prove that the sequence $a_n$ with

$$a_n = \sum_{k=1}^{n} \cos(kx)$$

is bounded if and only if $x$ is not a multiple of $2\pi$.

- Answer: If $x$ is a multiple of $2\pi$ then the all terms of the sum are 1, so $a_n = n$, and the sequence diverges.

On the other hand, if $x$ is not a multiple of $2\pi$ we will show that the sequence is bounded. In fact, we have $\cos kx = \Re\{e^{ikx}\} = \text{real part of } e^{ikx}$, hence

$$a_2 = \Re\left\{ \sum_{k=1}^{n} e^{ikx} \right\} = \Re\left\{ \frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1} \right\}.$$ 

If $x$ is not a multiple of $2\pi$ then the denominator is not zero, and

$$|a_n| \leq \left| \frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1} \right| \leq \frac{|e^{i(n+1)x}| + |e^{ix}|}{|e^{ix} - 1|} = \frac{2}{|e^{ix} - 1|},$$

hence the sequence is bounded, Q.E.D.
Problem A3. For certain $n \times n$-matrices $A$ and $B$, it is known that $AB = A + B$. Prove that $AB = BA$.

- Answer: If $I$ is the identity $n \times n$-matrix then we have:

$$(A - I)(B - I) = AB - A - B + I = AB - AB - I = I,$$

hence $B - I = (A - I)^{-1}$, and

$$I = (B - I)(A - I) = BA - B - A + I = BA - AB + I,$$

from which we get $BA - AB = 0$, and the desired result follows.
Problem A4. Determine whether the following statement is true or false. For every finite set $V$ of positive integers there exists a polynomial $P$ with integer coefficients such that $P(1/n) = n$ for all $n$ in $V$.

- Answer: It is true.

A way to find such polynomial is to notice that $xP(x) - 1$ must also be a polynomial with integer coefficients and roots at $1/n$ for $n \in V$. A polynomial with such property is $f(x) = \prod_{n \in V} (1 - nx)$, so $xP(x) - 1 = af(x)$ with $a$ integer would solve the problem. Note that the constant term of $f$ is $f(0) = 1$, hence we must take $a = -1$, and we get that the desired polynomial is $P(x) = \frac{1 - f(x)}{x} = \frac{1}{x} \left( 1 - \prod_{n \in V} (1 - nx) \right)$. 
Problem A5. Suppose that $a_n > 0$, and $\sum_{n=1}^{\infty} a_n$ converges. Show that there is a sequence \{b_n\} such that $0 < b_n \to \infty$, and $\sum_{n=1}^{\infty} a_n b_n$ converges.

- Answer: Under the given hypotheses we have that the tail of the series $d_m = \sum_{n=m}^{\infty} a_n$ is a decreasing sequence tending to zero. For each $k \geq 1$ let $m_k$ be the minimum $m$ such that $d_m < 1/4^k$, and let $b_n = 1$ for $n < m_1$, $b_n = 2^k$ for every $n$ such that $m_k \leq n < m_{k+1}$. Then $0 < b_n \to \infty$, and\(^1\)

$$
\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{m_1-1} a_n b_n + \sum_{k=1}^{\infty} \sum_{n=m_k}^{m_{k+1}-1} a_n b_n
$$

$$
= \sum_{n=1}^{m_1-1} a_n + \sum_{k=1}^{\infty} \left( 2^k \sum_{n=m_k}^{m_{k+1}-1} a_n \right)
\leq \sum_{n=1}^{m_1-1} a_n + \sum_{k=1}^{\infty} \frac{1}{2^k}
$$

$$
= \sum_{n=1}^{m_1-1} a_n + 1,
$$

hence $\sum_{n=1}^{\infty} a_n b_n$ also converges, Q.E.D.

\(^1\)Note: Empty sums (with no terms) have value zero, e.g., if $m_k = m_{k+1}$, then $\sum_{n=m_k}^{m_{k+1}-1} a_n$ is empty and its value is zero.
Problem A6. Let $a$, $b$, $c$ the side lengths of a triangle $T$. Prove that there is a triangle with side lengths $a^2$, $b^2$, and $c^2$ if and only if $T$ is acute (all its angles are acute).

- Answer: A necessary and sufficient condition for three positive numbers $x$, $y$ and $z$ to be side lengths of some triangle is $x < y + z$, $y < z + x$, and $z < x + y$. In our case that leads to $a^2 < b^2 + c^2$ and similar inequalities obtained by rotation of $a$, $b$, $c$. By the law of cosines applied to triangle $T$ we have $a^2 = b^2 + c^2 - 2bc \cos A$, where $A$ is the angle opposite to $a$. Then the condition $a^2 < b^2 + c^2$ is equivalent to $\cos A > 0$, or $A < \frac{\pi}{2}$ ($A$ acute). The same reasoning applies to the other angles.