Problem A1. Find all integer solutions to the system of equations
\[
\begin{cases}
  x^2 - y^2 = 16 \\
  x^3 - y^3 = 98
\end{cases}
\]

- Answer: We have \(x^2 - y^2 = (x - y)(x + y)\), and \(x^3 - y^3 = (x - y)(x^2 + xy + y^2)\) hence \(x - y\) is a common factor of 16 and 98, i.e., a divisor of \(\gcd(16, 98) = 2\). The second equation \(x^3 - y^3 = 98\) shows that \(x > y\), hence \(x - y > 0\), so the only possibilities are the positive divisors of 2, i.e., \(x - y = 1\) and \(x - y = 2\).

If \(x - y = 1\), then \(x + y = 16\), and from here we get \(x = 17/2, \ y = 15/2\), which are not integers and do not satisfy \(x^3 - y^3 = 98\).

If \(x - y = 2\), then \(x + y = 8\), and from here we get \(x = 5, \ y = 3\), which do satisfy the system. So, the only integer solution is \( (x, y) = (5, 3) \).
Problem A2. Prove that 48 divides $n^4 - 1$ if $n$ is not a multiple of 2 or 3.

- **Answer:** Since $48 = 16 \cdot 3$, it suffices to prove that (with the given conditions) 3 and 16 both divide $n^4 - 1$.

If $n$ is not a multiple of 3, then $n \equiv \pm 1 \pmod{3}$, and $n^4 \equiv 1 \pmod{3}$, hence $n^4 - 1 \equiv 0 \pmod{3}$, and $n^4 - 1$ is divisible by 3.

If $n$ is not a multiple of 2 then $n$ is odd, $n + 1$ and $n - 1$ are two consecutive even numbers, so one of them is also a multiple of 4, and $n^2 - 1 = (n + 1)(n - 1)$ is a multiple of 8.

On the other hand $n^2 + 1$ is even.

Consequently, $n^4 - 1 = (n^2 + 1)(n^2 - 1)$ is a multiple of $2 \cdot 8 = 16$.

Combined with the fact that $n^4 - 1$ is divisible by 3 we get that $n^4 - 1$ is divisible by 48, QED.
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Problem A3. We define a sequence \( \{a_n\} \) recursively in the following way: \( a_1 = 1 \), \( a_{n+1} = 2(a_n + 1) \) for \( n = 1, 2, 3, \ldots \). Find a close form for \( a_n \).

- Answer: We define a new sequence \( b_n = a_n + 2 \), which verifies \( b_1 = a_1 + 2 = 3 \), \( b_{n+1} = a_{n+1} + 2 = 2(a_n + 1) + 2 = 2a_n + 4 = 2(a_n + 2) = 2b_n \). Hence \( b_n = 3 \cdot 2^{n-1} \), and

\[
a_n = 3 \cdot 2^{n-1} - 2.
\]
Problem A4. Find a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \neq 0 \) and \( f(4x) = f(2x) + f(x) \) for every real \( x \).

- Answer: We try a function of the form \( f(x) = Ax^\alpha \), so it must verify \( A(4x)^\alpha = A(2x)^\alpha + Ax^\alpha \), or equivalently \( 4^\alpha = 2^\alpha + 1 \). From here we get \( 2^\alpha = \frac{1 \pm \sqrt{5}}{2} \). Only the positive solution makes sense in this case, and we get \( \alpha = \log_2 \phi \), where \( \phi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio. Hence an answer is \( f(x) = Ax^{\log_2 \phi} (A \neq 0) \), for \( x > 0 \).

Since the exponent is not an integer we must be careful about how to extend \( f \) to negative \( x \). The following extension works for every \( x \in \mathbb{R} \):

\[
 f(x) = \begin{cases} 
 Ax^{\log_2 \phi} & (x > 0) \\
 0 & (x = 0) \\
 B|x|^{\log_2 \phi} & (x < 0) 
\end{cases}
\]

where \( A, B \) do not need to be the same, but they should not be both zero if we want \( f \neq 0 \).

(Note 1: The same solution can be expressed in other ways using the identity \( x^{\log_2 \phi} = \phi^{\log_2 x} \).)

(Note 2: There are other possible solutions, e.g. \( f(x) = F_n \) for \( x \in [2^n, 2^{n+1}) \) \((n \in \mathbb{Z})\), where \( F_n \) is the bidirectional Fibonacci sequence, and \( f(-x) = f(x) \).)
Problem A5. Let $A, B, C, D$ be the following matrices:

$$
A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
$$

Is it possible to obtain the following matrix:

$$
E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

by multiplying the given matrices $A, B, C, D$ (in any order and any number of times)?

- Answer: The answer is no. If $P$ and $Q$ are two square matrices of the same dimension then $\det (PQ) = \det P \det Q$. We have $\det A = \det B = \det C = \det D = 1$, so any product of those matrices will have determinant 1. But $\det E = -1$. □
Problem A6. Let \( x, y, z \) be three real numbers such that \( 0 < x, y < \pi \), \( z = (x + y)/2 \). Prove:

\[
\sqrt{\frac{\sin x \sin y}{x \quad y}} \leq \frac{\sin z}{z}.
\]

- Answer: Let \( f : (0, \pi) \to \mathbb{R} \) the function \( f(x) = \ln(\frac{\sin x}{x}) = \ln \sin x - \ln x \). We will prove \( f(\frac{x+y}{2}) \geq \frac{1}{2}(f(x) + f(y)) \).

The derivative of \( f \) is \( f'(x) = \cot x - \frac{1}{x} \), and the second derivative is \( f''(x) = -\frac{1}{\sin^2 x} + \frac{1}{x^2} \). Since \( x > \sin x \) for \( x \in (0, \pi) \) we have \( f''(x) < 0 \) for every \( x \in (0, \pi) \), hence \( f \) is a concave function. As such it verifies \( \lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y) \) for \( x, y \in (0, \pi) \) and \( 0 \leq \lambda \leq 1 \). The result is the particular case \( \lambda = 1/2 \).

Finally, we have:

\[
\sqrt{\frac{\sin x \sin y}{x \quad y}} = \sqrt{e^{f(x)}e^{f(y)}} = e^{\frac{1}{2}(f(x)+f(y))} \leq e^{f(\frac{x+y}{2})} = e^{f(z)} = \frac{\sin z}{z}.
\]

\[\square\]