Problem A1. Show that
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(1+n)}} \leq \pi. \]
Problem A2. Find the following infinite product:

\[ P = \prod_{n=1}^{\infty} \left( 1 + \left( \frac{1}{7} \right)^{2n} \right) \]

Write the result as a fraction \( P = \frac{a}{b} \) in least terms.
Problem A3. Let $S$ be a set with even number of elements, and $f : S \to S$ a map of $S$ into itself such that $f \circ f : S \to S$ is the identity map. Show that the set of the fixed points has even number of elements.
Problem A4. Let \( f: \mathbb{R} \to \mathbb{R} \) a continuous function without fixed points, i.e., there is no \( x \in \mathbb{R} \) such that \( f(x) = x \). Let \( n \) be a positive integer. Prove that \( f^n = f \circ f \circ \cdots \circ f \) has no fixed points either.
Problem A5. The Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, \ldots are defined as \( F_0 = 0, \quad F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) (for \( n \geq 2 \)). The *digital root* of a non-negative integer is the (single digit) value obtained by an iterative process of summing digits, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached. For example, the digital root of 65,536 is 7, because \( 6 + 5 + 5 + 3 + 6 = 25 \) and \( 2 + 5 = 7 \). Prove that there are integers \( a, b \), with \( a > 0 \) and \( b \geq 0 \), such that all Fibonacci numbers of the form \( F_{an+b} \), \( n = 0, 1, 2, 3, \ldots \), have the same digital root.
Problem A6. Let $a, b, c$ three positive real numbers prove:

$$\sqrt{a^2 + 1} + \sqrt{b^2 + 4} + \sqrt{c^2 + 9} \geq 2\sqrt{3}\sqrt{a + b + c}.$$