Étale Homotopy Theory

and

Simplicial Schemes

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Cohomology and K-Theory

Singular Cohomology	*and*	Topological K-Theory
Étale Cohomology	*and*	Étale K-Theory
Motivic Cohomology	*and*	Algebraic K-Theory

The last row will play an important role in this conference.

I will discuss the second row.

Étale Cohomology

Let X be a scheme of finite type over a field k and let ℓ be a prime $\neq$ the characteristic of k.

The étale cohomology groups

$$H^p_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$$

can be defined "topologically" via the Čech construction. Let $\mathcal{U} = \{U_i \to X\}$ be an étale cover of X and set

$$U_{i_0,\ldots,i_p} = U_{i_0} \times_X \cdots \times_X U_{i_p}.$$

Then the étale p-chains are given by

$$C^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z}) = \prod H^0(U_{i_0,\ldots,i_p}, \mathbb{Z}/\ell\mathbb{Z}).$$

If we define

$$H^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z}) = H^p(C^*(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z})),$$

then the pth étale cohomology group of X is the direct limit

$$H^p_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z})$$

over the directed set of all étale covers $\mathcal{U}$ of X.

Key Observation: The global sections $H^0(U_{i_0,\ldots,i_p}, \mathbb{Z}/\ell\mathbb{Z})$ are determined by the set of connected components

$$\pi_0(U_{i_0,\ldots,i_p}).$$

As we vary over all $i_0, \ldots, i_p$, we get a *simplicial set.*

Simplicial Sets and Schemes

Let Δ be the category with objects $[n] = \{0, 1, \ldots, n\}$ and morphisms monotone maps $[n] \to [m]$.

A *simplicial object* in a category C is a contravariant functor

$$X_\bullet : \Delta \to C.$$

The maps $[1] \to [0]$ and $[0] \rightrightarrows [1]$ give

$$X_0 \leftleftarrows\!\!\!\rightarrow X_1 \cdots$$

$C = \text{Sets}$ gives SSets and $C = \text{Sch}/k$ gives SSch/k. We also have a connected component functor

$$\pi_0 : \text{SSch}/k \longrightarrow \text{SSets}.$$

Étale Homotopy Theory

Due to Artin and Mazur, using ideas of Verdier and Lubkin. $\mathcal{H}$ is the homotopy category of SSets (ignore base points).

Given a scheme X, $X_{\text{ét}}$ is the category with objects étale maps $Y \to X$ and morphisms commutative diagrams

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

where $Y \to Y'$ is also étale. By the Čech construction, an étale cover $\{U_i \to X\}$ gives a simplicial object $U.$ in $\mathrm{S}X_{\text{ét}}$. This is an example of a *hypercovering*.

The *étale homotopy type of* X

$$(X)_{\text{ét}} = \{\pi_0(U.)\} \in \text{Pro-}\mathcal{H}$$

given by the connected components of the inverse system of all hypercoverings of X.

If $X.$ is a simplicial scheme, one also has

$$(X.)_{\text{ét}} = \{\pi_0(\Delta U..)\} \in \text{Pro-}\mathcal{H}.$$

Furthermore, if $U.$ is a hypercovering of X, then the natural map

$$(U.)_{\text{ét}} \to (X)_{\text{ét}}$$

is a weak equivalence in Pro-$\mathcal{H}$.

Applications

Étale homotopy theory has many applications, including:

- Comparison Theorems
- The Adams Conjecture
- Tubular Neighborhoods
- Poincaré Duality
- Finite Chevalley Groups
- Étale K-Theory

Comparison Theorems

When X is a scheme of finite type over $\mathbb{C}$, the most basic comparison theorem asserts

$$H^p_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z}) \simeq H^p(X(\mathbb{C}), \mathbb{Z}/\ell\mathbb{Z})$$

for any prime ℓ. This generalizes:

- For X geometrically unibranch over $\mathbb{C}$,

  $$(X)_{\text{ét}} \simeq X(\mathbb{C})\hat{}$$

  in Pro-$\mathcal{H}$ ($\hat{}$ is pro-finite completion).

- For $X.$ over $\mathbb{C}$, we have

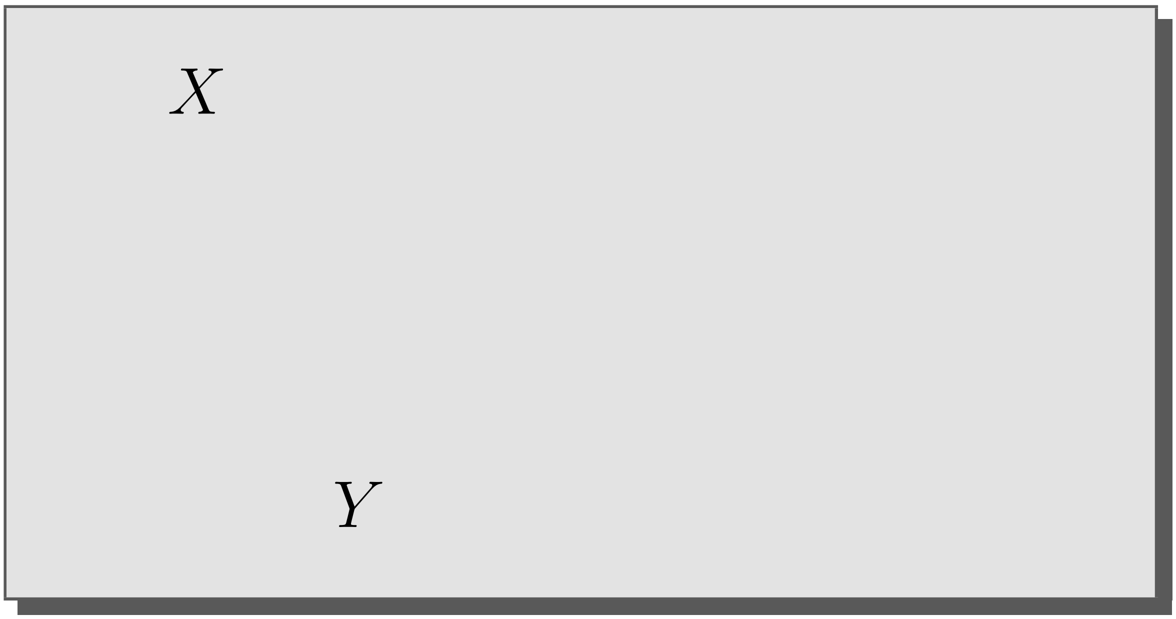
  $$(X.)_{\text{ét}}\hat{} \simeq_{weak} X.(\mathbb{C})\hat{}.$$

- For $f : X \to Y$ smooth and proper,

  $$H^p(\text{fib}(f_{\text{ét}}), \mathbb{Z}/\ell\mathbb{Z}) \simeq H^p_{\text{ét}}(f^{-1}(y), \mathbb{Z}/\ell\mathbb{Z}).$$

Tubular Neighborhoods

Topologically, a tubular neighborhood $T_{X/Y}$ of $Y \subset X$ is easy to picture:



Some nice properties of $T_{X/Y}$:

- $Y \subset T_{X/Y}$ is a homotopy equivalence.
- For X, Y smooth, $\partial T_{X/Y} \to Y$ is a spherical fibration that carries the Thom class. Up to homotopy, this fibration is $T_{X/Y} - Y \hookrightarrow T_{X/Y}$.

Tubular Neighborhoods in Algebraic Geometry

Zariski: A Zariski neighborhood of $Y \subset X$ is too big. Except in trivial cases, it can't be a tubular neighborhood.

Étale: An étale neighborhood is an étale map $V \to X$ such that $V \times_X Y \simeq Y$. These are also too big:

Example. One can prove that the only étale neighborhoods of

$$\mathbb{P}^1 \subset \mathbb{P}^2$$

are Zariski neighborhoods of $\mathbb{P}^1$ in $\mathbb{P}^2$.

Ringed Space: Given $Y \subset X$, one can construct:

- its *henselization* $Y \subset X^h_Y \to X$.
- its *formal completion* $Y \subset \hat{X}_Y \to X$.

These are ringed spaces supported on Y with some nice properties. But we can't remove Y to get a spherical fibration. So these aren't geometric enough.

Simplicial: Let $t_{X/Y}$ be the category of simplical objects $V. \in \mathrm{S}X_{\text{ét}}$ such that $V. \times_X Y \to Y$ is a hypercovering. Then:

The *tubular neighborhood* of Y in X is

$$T_{X/Y} = \{V. \mid V. \in t_{X/Y}\}.$$

Here is a glimpse of life before TeX:

In 1974, I paid $3 to have this page typed.

Properties of $T_{X/Y}$

- $(Y)_{\text{ét}} \simeq (T_{X/Y})_{\text{ét}}$ is a homotopy equivalence.
- $H^*_{\text{ét,Y}}(X, \mathbb{Z}/\ell\mathbb{Z})$ is isomorphic to $H^*(T_{X/Y}, T_{X/Y} - Y, \mathbb{Z}/\ell\mathbb{Z})$.
- When Y and X are smooth, there is an algebraic exponential map

$$(N_{X/Y} - Y)_{\text{ét}}\hat{} \simeq (T_{X/Y} - Y)_{\text{ét}}\hat{}$$

  where $N_{X/Y}$ is the normal bundle of Y in X.
- Friedlander used $T_{X/Y}$ to give a "topological" proof of Poincaré duality for étale cohomology.

Twisted Chevalley Groups

Let H be a "twisted" group of Chevalley, Steinberg, or Suzuki-Rees type. Then there is a simple algebraic group G over $\overline{\mathbb{F}}_p$ such that $H =$ the fixed point set of an algebraic endomorphism

$$\phi : G \longrightarrow G.$$

In 1953, Lang showed that $\Phi(g) = g\phi(g)^{-1}$ is onto. This gives a fibration

$$H \longrightarrow G \xrightarrow{\Phi} G.$$

In 1970 Quillen suggested that this would be relevant to étale homotopy theory. Friedlander pursued this in the 1970s. His results compute the $\mathbb{Z}/\ell\mathbb{Z}$ cohomology of H in terms of $H^*(BG, \mathbb{Z}/\ell\mathbb{Z})$ for $\ell \neq p$.

Classifying spaces were originally constructed topologically and are not algebraic varieties.

Working simplicially, we have the simplicial scheme BG such that BG_n is the cartesian product

$$\underbrace{G \times_k \cdots \times_k G}_{n \text{ times}}.$$

Boundary and degeneracy maps are built from the identity $\mathrm{Spec}(k) \to G$ and multiplication $G \times_k G \to G$.

By the comparison theorem, the étale homotopy type of BG is the same as $BG(\mathbb{C})$, up to pro-finite completion. This brings topology into algebraic geometry.

Étale K-Theory

For a CW complex T, ordinary K-theory with coefficients in $\mathbb{Z}/m\mathbb{Z}$ is defined by

$$\begin{aligned} K^0(T, \mathbb{Z}/m\mathbb{Z}) &= [C(m) \wedge T, BU] \\ K^1(T, \mathbb{Z}/m\mathbb{Z}) &= [\Sigma C(m) \wedge T, BU] \end{aligned}$$

where $C(m)$ comes from the cofiber triple

$$S^1 \xrightarrow{m} S^1 \longrightarrow C(m).$$

Using étale homotopy theory, we get the following definition of Friedlander:

The *étale K-theory* of a scheme X is

$$K^0_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}) = [C(m) \wedge (X)_{\text{ét}}, \#BU]$$
$$K^1_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}) = [\Sigma C(m) \wedge (X)_{\text{ét}}, \#BU]$$

Properties

- $K^*_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z}) \simeq K^*(X(\mathbb{C}), \mathbb{Z}/\ell\mathbb{Z})$.
- $\text{Gal}(\overline{k}/k)$ acts on $K^*_{\text{ét}}(X \times_k \overline{k}, \mathbb{Z}/\ell\mathbb{Z})$.
- There is a spectral sequence relating $H^*_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$ to $K^*_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$.
- The map $K^0_{\text{alg}}(X) \to K^0(X(\mathbb{C})) \otimes \mathbb{Z}_\ell$ factors through $K^*_{\text{ét}}(X, \mathbb{Z}_\ell)$.

A more sophisticated definition of étale K-theory was given in 1985 by Dwyer and Friedlander. Most cohomology theories can be represented by *spectra*. Just as we brought the topological BG into the category of simplicial schemes, the idea here is to bring spectra into SSch/k.

First: Given a k-algebra A,

$$\mathbf{K}_A = \mathrm{Sp}(\mathrm{Hom}_g(A, B\overline{\mathcal{G}\ell_*})_k),$$

where "g" means scheme-theoretic maps. Then

$$\pi_i(\mathbf{K}_A) = \text{Quillen } K\text{-theory of } A.$$

Second: Given a scheme X over k,

$$\hat{\mathbf{K}}_X^{\text{ét}} = \mathrm{Sp}(\mathrm{Hom}_l(X, B\overline{\mathcal{G}\ell_*})_k),$$

where "ℓ" means maps between the ℓ-adic completions of the étale homotopy types. Then

$$\pi_i(\hat{\mathbf{K}}_X^{\text{ét}} \wedge \mathcal{M}(\nu)) = \hat{K}_i^{\text{ét}}(X, \mathbb{Z}/\ell^\nu\mathbb{Z}).$$