Support varieties for Weyl modules

Northwestern University
Friedlander conference.

I. Support varieties for Lie algebras

based on Friedlander - Parshall, Inv. Math 1986.

$k = \overline{\mathbb{F}_p}$ algebraically closed field.
$(g, [p])$ restricted Lie algebra

$u(g) = U(g) / (x^p - x^{[p]})$ restricted enveloping algebra
(finite-dim cocomm Hopf alg)

$R = H^{2\bullet}(u(g), k)$ finitely generated comm k-algebra

Let $M \in \mathrm{mod}(u(g))$

$$\boxed{V_g(M) = \mathrm{Maxspec}\left(R/J_M \right) \qquad J_M = \mathrm{Ann}_R \, \mathrm{Ext}^{\bullet}_{u(g)}(M, M)}$$

(support variety of the module M)

Basic facts

1) $V_{\mathfrak{g}}(M) \subseteq V_{\mathfrak{g}}(k)$ for all $M \in \mathrm{mod}(u(\mathfrak{g}))$

2) $V_{\mathfrak{g}}(k) = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$ (Jantzen)

3) $V_{\mathfrak{g}}(M) = \{x \in \mathfrak{g} \mid x^{[p]} = 0,\ M|_{\langle x \rangle} \text{ is not free}\} \cup \{0\}$

(F-P, Suslin-F-Bendel)

- Although, support varieties can be characterized in this way, they are often difficult to compute.

<u>Open question (F-P)</u>

When is $V_{\mathfrak{g}}(k)$ an irreducible variety?

I Reductive groups

→ SLIDE WITH NOTATION.

Connection with conjugacy classes.

Let M be a rational G-module.

$$V_{\mathfrak g}(M) \subseteq V_{\mathfrak g}(k) = \{ x \in \mathfrak g \mid x^{[p]} = 0 \} \subseteq \mathcal N$$

(restricted nullcone) (nullcone)

G-stable.

- orbits of $\mathcal N$ under G are classified
- $V_{\mathfrak g}(M) = \overline{G\cdot x_1} \cup \dots \cup \overline{G \cdot x_s}$

Weyl modules

Let $\lambda \in X(T)_+$, set $H^0(\lambda) = \operatorname{ind}_B^G \lambda = \mathcal H^0(G/B, \mathcal L(\lambda))$

$V(\lambda) = H^0(-w_0\lambda)^*$ Weyl module

Fact: $\boxed{V_{\mathfrak g}(V(\lambda)) = V_{\mathfrak g}(H^0(\lambda))}$

Question: Determine $V_{\mathfrak g}(H^0(\lambda))$.

Good versus Bad Primes

- $\Phi = A_n$ all primes
- $\Phi = B_n, C_n, D_n$ $p > 2$
- $\Phi = E_6, E_7, F_4, G_2$ $p > 3$
- $\Phi = E_8$ $p > 5$

} GOOD.

- all other primes (rel to Φ) are BAD.

III Jantzen conjecture (p-good)

Let $\lambda \in X(T)$

$$\Phi_\lambda = \{\alpha \in \Phi \mid (\lambda+\rho, \alpha^\vee) \in p\mathbb{Z}\}$$

p-good $\Rightarrow$ Φ_λ is a subroot system of Φ.
$\exists\, w \in W$ $w(\Phi_\lambda) = \Phi_J$ $J \subseteq \Delta$.

With $J \subseteq \Delta$ $\mathfrak{g} = \mathfrak{u}_J \oplus \mathfrak{l}_J \oplus \mathfrak{u}_J^+$.

<u>Thm</u>: (N, Parshall, Vella) Let G be a reductive grp., p-good.
Let $\lambda \in X(T)_+$. with $w \in W$ $w(\Phi_\lambda) = \Phi_J$ $J \subseteq \Delta$.
Then

$$V_{\mathfrak{g}}(H^0(\lambda)) = G \cdot \mathfrak{u}_J.$$

STATE

- This theorem was conjectured by Jantzen 1987, who prove it for type A.
- $\overline{G \cdot u_J}$ is the closure of a Richardson orbit.

<u>Corollary</u>: Let p be good. Then $\mathcal{N}_1$ is an irreducible variety

Pf $\mathcal{N}_1 = V_{\mathfrak{g}}(k) = G \cdot u_J \quad J \subseteq \Delta$.

Carlson, Lin, N, Pearsball [CLNP] (2003) completely determined $\mathcal{N}_1$ using [NPV] for good primes.

STATE: Why is $\mathcal{N}_1$ important, evidence that this algebraic variety will be important for studying modules rep theory when $p < h$.

<u>Example:</u> Compute $\mathcal{N}_1$ for $\Phi = E_7$ $p = 13$.

1) $W(\Phi_0) = \Phi_J$

2) $\dim \mathcal{N}_1 = \dim G \cdot u_J = |\Phi| - |\Phi_J| = |\Phi| - |\Phi_0|$

3) $\Phi_0 = \{ \alpha \in \Phi \mid (\rho, \alpha^\vee) \in 13\mathbb{Z} \}$
(roots of height 13)

$\Phi_0 = A_1 \times A_1$

$\dim \mathcal{N}_1 = 126 - (2+2) = 122.$

$\mathcal{N}_1 = \overline{\mathcal{O}(E_7(a_2))}$ (only orbit of dim 122)

→ SLIDE WITH EXCEPTIONAL CASES

IV Work with UGA VIGRE Algebra.

- 2003-04. UGA VIGRE Algebra Group.

- 5 faculty, 4 postdocs, 7 students } led by D Benson, B. Boe, me.

- produced 3 papers

→ SHOW SLIDE.

<u>Setting</u>

Let $\rho: G \longrightarrow GL(V)$ be a representation
st $\ker \rho \cap \mathcal{U} = \{e\}$

$\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ with $\ker \rho \cap \mathcal{N} = \{0\}$.

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid \rho(x)^r = 0\}$$

<u>Problem</u>: Describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as the union of orbit closures.

<u>Highlights</u>

1) For $k = \mathbb{C}$ or $k = \overline{\mathbb{F}_p}$ p good., we calculated $\mathcal{N}_{r,e}(\mathfrak{g})$ for ρ = adjoint, min.

2) As a corollary, set $r = p$ $k = \overline{\mathbb{F}_p}$.

$$\mathcal{N}_1 = \mathcal{N}_{p,e}(\mathfrak{g})$$

This gives a new (and simpler) method to compute the restricted nullcone. (without using the Jantzen conj)

3) When $k = \overline{\mathbb{F}_p}$ p - bad., we computed $\mathcal{N}_1 = \mathcal{N}_1(\mathfrak{g})$ completely.

→ SHOW SLIDE (thm, ~~Hasse diagrams~~)

- $\mathcal{N}_1$ need not be the closure of a Richardson orbit.

- $\mathcal{N}_1$ is <u>still</u> an irreducible variety

4) In the process we had to generate a complete list of orbit reps. for exceptional Lie algebras and all primes.

5) Let $\mathcal{U}$ be the unipotent variety.

$k = \mathbb{C}$ $\quad\quad \mathcal{N} \cong \mathcal{U}$ $\quad\quad$ via exp map

$k = \mathbb{F}_p$ $\quad$ p good $\quad\quad \mathcal{N} \cong \mathcal{U}$ $\quad\quad$ via Springer iso.

$k = \mathbb{F}_p$ $\quad$ p bad $\quad\quad \mathcal{N} \not\cong \mathcal{U}$ $\quad\quad$ different # of orbits.

Set $\mathcal{U}_1 = \{u \in \mathcal{U} \mid u^p = 1\}$

<u>Theorem</u> (UGA VIGRE) Let G be a connected simple alg group. There exists a 1-1 correspondence between orbits in $\mathcal{N}_1$ and $\mathcal{U}_1$ which respects the closure ordering of orbits.

STATE

- asked several experts (Carter, Kazhdan, Testerman) they were not aware of this.
- Is there a G-equivariant map between $\mathcal{N}_1$ and $\mathcal{U}_1$.

V Support varieties of Weyl module (bad primes)

<u>Theorem</u>: $\dim V_{\mathfrak{g}}(H^0(\lambda)) = |\Phi| - |\Phi_\lambda|$ for all p.

We proved this theorem and computed $V_{\mathfrak{g}}(H^0(\lambda))$ for $k = \overline{\mathbb{F}}_p$ p - bad.

- $V_{\mathfrak{g}}(H^0(\lambda))$ is always irreducible, need not be Richardson.
- done by computer calculation (Christkofosky, + students)

E_8 $p = 5$ calculation

→ SLIDE

Notation

G semisimple or reductive algebraic group, $\mathfrak{g} = \operatorname{Lie} G$

B Borel subgroup, $\mathfrak{b} = \operatorname{Lie} B$

U unipotent radical of B, $\mathfrak{u} = \operatorname{Lie} U$

T maximal torus

Φ root system

Δ simple roots

W Weyl group

h Coxeter number

ρ half sum of positive roots

$X(T)$ integral weights

$X(T)_+$ dominant integral weights

$X_1(T)$ restricted weights

Type E_6:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	62	$A_2 \times A_1 \times A_1$	$\{1,2,4,6\}$	$A_4 + A_1$
7	66	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_6(a_3)$
11	70	A_1	$\{4\}$	$E_6(a_1)$
≥ 13	72	$\emptyset$	$\emptyset$	E_6

Type E_7:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	106	$A_3 \times A_2 \times A_1$	$\{1,2,3,5,6,7\}$	$A_4 + A_2$
7	114	$A_2 \times A_1 \times A_1 \times A_1$	$\{1,2,3,5,7\}$	A_6
11	120	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_7(a_3)$
13	122	$A_1 \times A_1$	$\{4,6\}$	$E_7(a_2)$
17	124	A_1	$\{4\}$	$E_7(a_1)$
≥ 19	126	$\emptyset$	$\emptyset$	E_7

Type E_8:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
7	212	$A_4 \times A_2 \times A_1$	$\{1,2,3,5,6,7,8\}$	$A_6 + A_1$
11	224	$A_2 \times A_2 \times A_1 \times A_1$	$\{1,2,3,5,6,8\}$	$E_8(a_6)$
13	228	$A_2 \times A_1 \times A_1 \times A_1$	$\{2,3,5,6,8\}$	$E_8(a_5)$
17	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{2,3,5,7\}$	$E_8(a_4)$
19	234	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_8(a_3)$
23	236	$A_1 \times A_1$	$\{4,6\}$	$E_8(a_2)$
29	238	A_1	$\{4\}$	$E_8(a_1)$
≥ 31	240	$\emptyset$	$\emptyset$	E_8

Type F_4:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	40	$A_2 \times A_1$	$\{1,3,4\}$	$F_4(a_3)$
7	44	$A_1 \times A_1$	$\{1,3\}$	$F_4(a_2)$
11	46	A_1	$\{3\}$	$F_4(a_1)$
≥ 13	48	$\emptyset$	$\emptyset$	F_4

Type G_2:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	10	A_1	$\{2\}$	$G_2(a_1)$
≥ 7	12	$\emptyset$	$\emptyset$	G_2

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Restricted Nullcone for Bad Primes

Theorem:

Let G be a simple classical connected algebraic group over k where $\operatorname{char} k = 2$. For each type $\Phi = B_l$, C_l and D_l the following holds.

(i) If Φ is of type B_l then $\mathcal{N}_1 = \overline{\mathcal{O}(2_2^l, 1_1)}$.

(ii) If Φ is of type C_l then $\mathcal{N}_1 = \overline{\mathcal{O}(2_1^l)}$.

(iii) If Φ is of type D_l then $\mathcal{N}_1 = \overline{\mathcal{O}(2_2^l)}$.

Theorem:

Let G be an exceptional algebraic group with p a bad prime.

(i) If Φ is of type E_6 then $\mathcal{N}_1 = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ $(p = 3)$, $3A_1$ $(p = 2)$.

(ii) If Φ is of type E_7 then $\mathcal{N}_1 = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.

(iii) If Φ is of type E_8 then $\mathcal{N}_1 = \overline{\mathcal{O}(X)}$ where $X = A_4 + A_3$ $(p = 5)$, $2A_2 + 2A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.

(iv) If Φ is of type F_4 then $\mathcal{N}_1 = \overline{\mathcal{O}(X)}$ where $X = A_1 + \widetilde{A_2}$ $(p = 3)$, $A_1 + \widetilde{A_1}$ $(p = 2)$.

(v) If Φ is of type G_2 then $\mathcal{N}_1 = \overline{\mathcal{O}(X)}$ where $X = G_2(a_1)$ $(p = 3)$, $\widetilde{A_1}$ $(p = 2)$.

E8, p=5

$|W\cdot\lambda|$ $\qquad$ $V_g(H^0(\lambda))$

λ	$\|O\|$	$\|\Phi\|-\|\Phi_\lambda\|$	**Orbit Type**
(4,4,4,4,4,4,4,4)	1	0	O
(0,4,4,4,4,4,4,4)	2,160	156	$2A_2$
(4,0,4,4,4,4,4,4)	17,280	184	$D_4(a_1)+A_2$
(4,4,0,4,4,4,4,4)	69,120	196	$A_4+A_2+A_1$
(4,4,4,0,4,4,4,4)	48,384	200	A_4+A_3
(4,4,4,4,4,0,4,4)	60,480	194	A_4+A_2
(4,4,4,4,4,4,0,4)	6,720	166	$D_4(a_1)$
(4,4,4,4,4,4,4,0)	240	114	A_2
(3,3,3,3,3,3,2,1)	69,120	196	$A_4+A_2+A_1$
(3,3,3,3,3,2,3,2)	60,480	194	A_4+A_2
(3,3,3,3,0,4,4,4)	30,240	180	A_4
(4,4,4,4,4,4,4,1)	240	114	A_2
(4,4,4,4,4,4,0,0)	6,720	166	$D_4(a_1)$
(4,4,4,4,4,1,0,3)	2,160	156	$2A_2$
(4,4,4,4,0,1,3,4)	17,280	184	$D_4(a_1)+A_2$

$5^8 = 390{,}625 = |X_1(T)|$

structure of $\mathcal{N}_1$

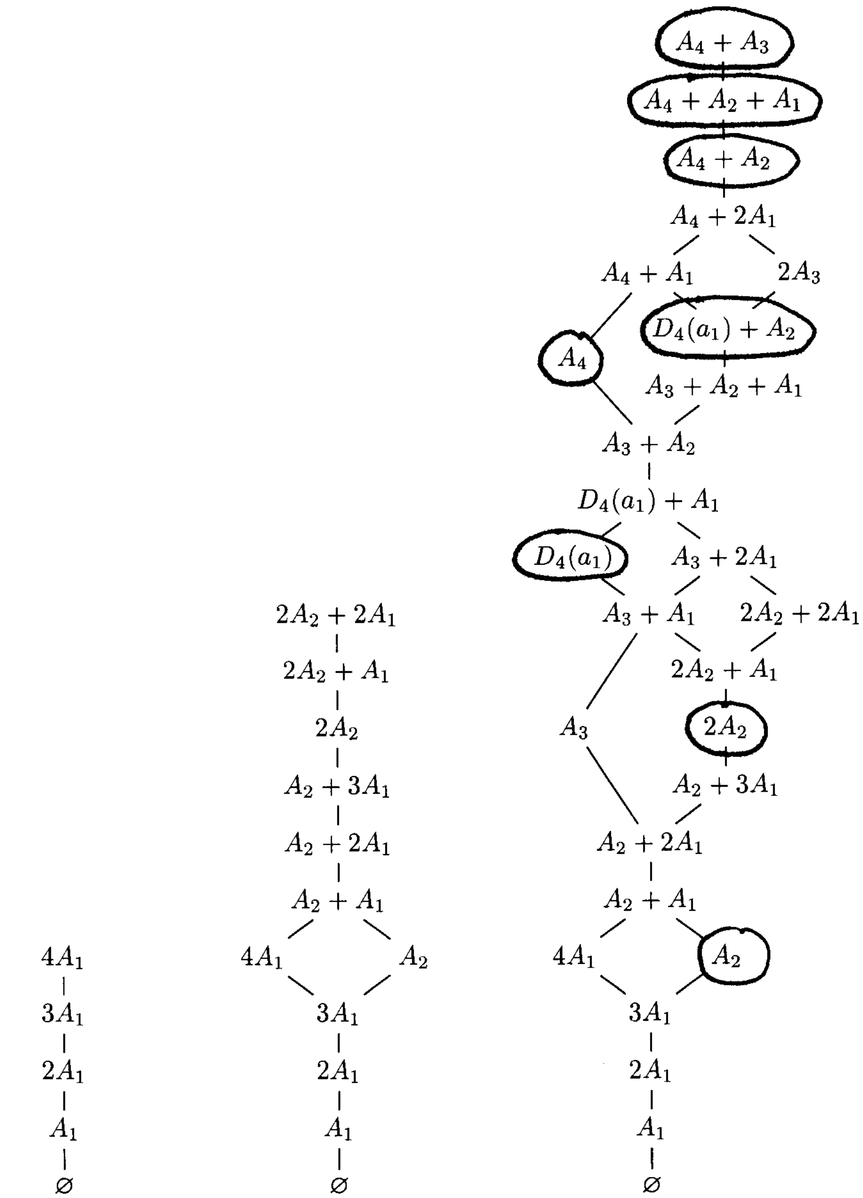

FIGURE 1. E_8 for $p = 2$ (left), $p = 3$, (center), and $p = 5$ (right)