Algebra Preliminary Examination

Northwestern University, September 2017

Do all of the following questions. Each question is worth 0.5 points.

Question 1. Let \( \alpha \) be a root of \( X^6 + X^3 + 1 \). Find all homomorphisms \( \mathbb{Q}(\alpha) \rightarrow \mathbb{C} \) of fields.

Question 2. Let \( S_3 \) be the symmetric group on 3 elements, and \( k \) an algebraically closed field of characteristic zero.
   1. Find all conjugacy classes of \( S_3 \).
   2. Find the dimension and the multiplicity in the regular representation of all irreducible representations of \( S_3 \) over \( k \).
   3. Write down the character table of \( S_3 \) over \( k \).

Question 3. Let \( K \) be a field and let \( M_3(K) \) denote the \( K \)-algebra of 3-by-3 matrices. Let \( B \) denote the subalgebra of \( M_3(K) \) of upper-triangular matrices. Determine whether \( B \) is semisimple.

Question 4. Put \( R = \mathbb{F}_q[X,Y]/\langle X^qY - XY^q \rangle \) where \( q \) is a power of prime. Let \( x,y \) be the image of \( X,Y \) in \( R \), respectively. Show that for every \( a \in \mathbb{F}_q \), \( R \) is not a finitely generated module over \( \mathbb{F}_q[y - ax] \).

Question 5. Let \( G \) be a group and \( H \) a subgroup of finite index. Show that there exists a normal subgroup \( N \) of \( G \) contained in \( H \) and also of finite index.

Question 6. Let \( R \) be a commutative ring that is a finitely generated \( \mathbb{Z} \)-algebra. Let \( \mathfrak{m} \) be a maximal ideal of \( R \). Show that \( R/\mathfrak{m} \) is a finite field.