**Preliminary Examination in Dynamical Systems**
**September, 2000**

1. Let $f : \mathbf{R} \to \mathbf{R}$ be an orientation-preserving contracting $C^1$ diffeomorphism (that is, assume that $0 < f'(x) < 1$, for all $x \in \mathbf{R}$). Prove that $f$ is topologically conjugate to the linear map $x \mapsto x/2$. Give an example that shows that the conjugating map need not be $C^1$.

2. Let $f(x) = \frac{1}{2}x + \frac{1}{8}x^3$ on $\mathbf{R}$. Prove that $f$ is structurally stable in terms of the $C^1$-sup topology on functions on $\mathbf{R}$.

3. Let $A$ be a transition matrix for a subshift of finite type, and let $\Sigma_A$ be the associated (2-sided) shift space.

(a) Prove that if there is an element $\mathbf{s} \in \Sigma_A$ that contains the symbol $i$ at least twice, then there is a periodic element $\mathbf{s}' \in \Sigma_A$ such that $s'_0 = i$. Such a symbol $i$ is called *essential*.

(b) Prove that any $\omega$-limit point in $\Sigma_A$ contains only essential symbols.

4. Let $f$ be a diffeomorphism and $\mathbf{p}$ a periodic point for $f$. Let $H_{\mathbf{p}}$ be the equivalence class of periodic points which are heteroclinically-related to $\mathbf{p}$. Let $\Lambda_{\mathbf{p}} = \overline{H_{\mathbf{p}}}$. Prove that $f$ is topologically transitive on $\Lambda_{\mathbf{p}}$.

5. Let $\mathbf{D}^2 = \{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 < 100\}$, and let $f : \mathbf{D}^2 \to \mathbf{D}^2$ be a $C^1$ diffeomorphism. Suppose that $\mathbf{p} = (0, 0)$ is a hyperbolic fixed point for $f$ and that

$$\mathbf{W}^u(\mathbf{p}) \supset \{(x, 0) \,|\, 0 \le x \le 5\}, \text{ and}$$

$$\mathbf{W}^s(\mathbf{p}) \supset \{(x, (x-4)^2) \,|\, 3 \le x \le 5\}.$$

Prove that for every $\epsilon > 0$, there is a diffeomorphism $g : \mathbf{D}^2 \to \mathbf{D}^2$ such that: (1) $d_{C^1}(f, g) < \epsilon$, and (2) $g$ has infinitely many periodic points.

6. Let $f$ be a diffeomorphism with a hyperbolic attracting set $\Lambda$. Prove that $\mathbf{W}^u(\mathbf{p}) \subset \Lambda$ for any $\mathbf{p} \in \Lambda$.

7. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a diffeomorphism and $\mathbf{p}$ a hyperbolic saddle fixed point. Assume $\mathbf{q}$ is a homoclinic point for $\mathbf{p}$.

(a) Prove that $\mathbf{q}$ is a chain recurrent point.

(b) Prove that $\mathbf{q}$ is a nonwandering point provided that $\mathbf{q}$ is a transverse homoclinic point for $\mathbf{p}$.