Preliminary Exam - Dynamical Systems (Fall 2001)

Do 7 out of the following first 8 problems

Problem 1. Describe the chain recurrent set of the function $f : R \to R$ given by $f(x) = 10x(1-x)$. Include a description of the dynamics of f restricted to the chain recurrent set and prove that this description is correct.

Problem 2. State the shadowing lemma. Consider the function $f : S^1 \to S^1$ given by $f(z) = z^3$ where z is a complex number of modulus 1. Suppose $\varepsilon = 1/100$ and $\{y_n\}_{n=-\infty}^{\infty} \subset S^1$ satisfies $|f(y_n) - y_{n+1}| < \varepsilon$. Prove there is a sequence $\{x_n\}_{n=-\infty}^{\infty}$ such that $f(x_n) = x_{n+1}$ and $|x_n - y_n| < \varepsilon$.

Problem 3. Define $\sigma : \Sigma_n \to \Sigma_n$, the full shift on n symbols.

1. Prove that the periodic points of σ are dense in Σ_n.
2. Prove that σ has a point with a dense forward orbit.
3. Prove that σ^k is topologically conjugate to σ if and only if $k = \pm 1$.

Problem 4. Let $f : X \to X$ be a homeomorphism on a connected compact metric space X. Show that

(a). f is topologically transitive if and only if every f invariant continuous function is a constant function.

(b). f is ergodic with respect to some invariant measure μ if and only if every f invariant integrable function (with respect to μ) is constant a.e. (μ).

Problem 5.

Let f be an anosov diffeomorphism on a compact manifold M and let $TM = E^s \oplus E^u$ be the continuous, f-invariant splitting. Assume that a diffeomorphism g of M commutes with f, i.e., $f \cdot g = g \cdot f$. Show that g preserves both E^s and E^u.

Problem 6.

Let $S \subset \Sigma_2$ be a subset of bi-infinite symbol sequences of 0 and 1s such that if $s = \{s_i\}_{i \in \mathbb{Z}} \in S$ and $s_i = 1$ then $s_{i+1} = s_{i+2} = 0$, i.e., every symbol 1 has to be followed by at least two 0s. Find the topological entropy of $\sigma : S \to S$.

Problem 7.

Let f be an orientation-preserving circle homeomorphism and $\tilde{f}$ be a lift of f. Show that for every irrational number α, there is a unique c such that rotation number $\rho(f + c) = \alpha$.

Problem 8.

Consider Hamiltonian function $H(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(q_1, \ldots, q_n)$. Assume that $q_1 = \ldots = q_n = 0$ is a nondegenerate maximum for V. Show that the origin is a hyperbolic saddle point for the Hamiltonian flow.

Optional

Let f be diffeomorphism of a compact, connected manifold M and let Λ be a closed hyperbolic invariant set for f. Assume that Λ contains a nonempty interior. Show that $\Lambda = M$, i.e., f is Anosov.