Dynamical Systems Prelim
September 20, 1991

1.

i) Define *topological conjugacy.*
ii) State Hartman's Theorem (also known as the Hartman–Grobman Theorem).
iii) Consider the pairs of linear functions from R^2 to R^2 given by the following matrices. For each pair tell whether the corresponding linear functions are topologically conjugate, linearly conjugate, both, or neither. Give reasons that they are or are not.

a) $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 100 \\ 0 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix}$

2. Define $\sigma : \Sigma_2 \to \Sigma_2$ the full two shift; i.e. define the metric space Σ including the metric on it, and define the function σ.
a) Prove that σ is a homeomorphism.
b) Prove that σ is topologically conjugate to σ^{-1} and that it is not topologically conjugate to σ^n for any $n > 1$.
c) Define the *zeta function* of a homeomorphism and calculate the zeta function for the above σ.

3. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 4x, & \text{if } x \leq 1/2; \\ 4 - 4x, & x \geq 1/2. \end{cases}$$

Define $J_0 = [0, 1/4]$ and $J_1 = [3/4, 1]$. Define

$$\Lambda = \{x | f^k(x) \in J_0 \cup J_1 \text{ for all } k \geq 0\}.$$

a) Sketch the graph of f and describe the behavior of $f^n(x)$ for large values of n if $x > 1$ or $x < 0$.
b) Define an itinerary function $s : \Lambda \to \Sigma_2$, where $\sigma : \Sigma_2 \to \Sigma_2$ is the one-sided shift map.
c) Prove that s is a homeomorphism.
d) Prove that $\sigma(s(x)) = s(f(x))$ for all $x \in \Lambda$.

4. Define *hypebolic invariant set* for a diffeomorphism $f : M \to M$ of a compact manifold M. State the *stable manifold theorem* for a hypebolic invariant set.

5. Define rotation number for a degree one homemorphism $f : S^1 \to S^1$.
   a) Prove that if two such homeomorphisms are topologically conjugate then they have the same rotation number.
   b) Give an example to show the converse of the statement in b) is false.
   c) Let $\phi_t(x)$ and $\psi_t(x)$ be the flows arising from the differential equations $y' = Ay$ and $y' = By$ for the matrices

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \textit{ and } B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

   where $0 < a < b$. Show that the time one maps $\phi_1(x)$ and $\psi_1(x)$ of these flows are topologically conjugate if and only if $a = b + 2\pi n$ for some integer n.

6. (a) Define *Markov partition* and *subshift of finite type.*
   (b) Construct a Markov partition for some smooth map of your choice, being sure to give the matrix of the associated subshift of finite type.