

Preliminary Examination in Dynamical Systems
September 1995

1. Find a topological conjugacy between the two linear isomorphisms induced by the following two matrices: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Show that there is no differentiable conjugacy between these two maps.
2. Assume $f : [0, 1] \rightarrow [0, 1]$ is continuous, and there exist two disjoint intervals I_1 and I_2 such that $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1$. Show that the topological entropy of f is at least $\log\left(\frac{1 + \sqrt{5}}{2}\right)$.
3. Assume that $\varphi^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous flow and the positive orbit of the point \mathbf{p} is bounded. Show that the ω -limit set of \mathbf{p} is invariant, closed and connected.
4. Assume $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^∞ Hamiltonian function with corresponding Hamiltonian differential equations $\dot{p} = -\frac{\partial H(p, q)}{\partial q}$, $\dot{q} = \frac{\partial H(p, q)}{\partial p}$. Assume (p_0, q_0) is a fixed point. Show that if λ is an eigenvalue of the linearization at (p_0, q_0) , then so is $-\lambda$.
5. (a) Define rotation numbers for circle homeomorphisms.
 (b) Show that the rotation number is rational if and only if the map has a periodic point.
6. Consider the map given in polar coordinates by

$$F \begin{pmatrix} \theta \\ r \end{pmatrix} = \begin{pmatrix} 8\theta - \frac{\pi}{8} \\ 1 + \frac{1}{16}r + \frac{2}{\pi}\theta \end{pmatrix}$$

for $0 \leq \theta \leq \frac{\pi}{2}$ and $1 \leq r \leq 2$. (Notice this definition is only for part of the plane.) Let

$$V_L = \{(\theta, r) : 0 \leq \theta \leq 3\pi/32, 1 \leq r \leq 2\}$$

$$V_R = \{(\theta, r) : \pi/4 \leq \theta \leq 11\pi/32, 1 \leq r \leq 2\}.$$

Prove that the maximal invariant set in $V_L \cup V_R$ is hyperbolic.

7. Assume Λ is a compact isolated hyperbolic invariant set for a diffeomorphism f on \mathbb{R}^n . Prove that Λ has a local product structure.