Preliminary Examination in Dynamical Systems
February 1996

1. Let $f_\lambda : S^1 \to S^1$ be a continuous family of orientation-preserving circle homeomorphisms.
   (a) Define the rotation numbers.
   (b) Show that the rotation numbers of f_λ is continuous with respect to λ.

2. Assume $f : [0,1] \to [0,1]$ is continuous, and there exist two disjoint intervals I_1 and I_2 such that $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1$.
   (a) Show that there is a semi-conjugacy onto a subshift of finite type.
   (b) Show that the topological entropy of f is at least $\log\big(\frac{1+\sqrt{5}}{2}\big)$.

3. Let $\varphi^t : R^n \to R^n$ be a smooth flow. Assume that the positive orbit of a point $p \in R^n$ is bounded. Show that the distance between $\varphi^t(p)$ and $\omega(p)$ approaches to zero as t approaches ∞.

4. Consider the periodic differential equation on R: $x' = f(x,t)$, where $f(x, t+1) = f(x)$. Let $\varphi^t(x)$ be a solution of the equation such that $\varphi^0(x) = x$.
   (a) Show that the Poincaré map $\Pi = \varphi^1 : R \to R$ is monotonic, i.e., for any x, either $\Pi^i(x) \geq \Pi^{i-1}(x)$, for all $i = \ldots, -1, 0, 1, 2, \ldots$ or $\Pi^i(x) \leq \Pi^{i-1}(x)$, for all $i = \ldots, -1, 0, 1, 2, \ldots$.
   (b) Prove that the following equation has at least one periodic solution: $x' = -x^3 + \cos(t)$. (hint: show that every solution is positively bounded.)

5. Consider the map given in polar coordinates by

$$F\begin{pmatrix}\theta \\ r\end{pmatrix} = \begin{pmatrix} 8\,\theta - \dfrac{\pi}{8} \\ 2 + \dfrac{1}{16}\,r + \dfrac{1}{16}\,\sin(\theta) \end{pmatrix}$$

   for $0 \leq \theta \leq \dfrac{\pi}{2}$ and $1 \leq r \leq 3$. (Notice this definition is only for part of the plane.) Let

$$V_L = \{(\theta, r) :\ 0 \leq \theta \leq 3\,\pi/32,\ 1 \leq r \leq 3\}$$
$$V_R = \{(\theta, r) :\ \pi/4 \leq \theta \leq 11\,\pi/32,\ 1 \leq r \leq 3\}.$$

   Prove that the maximal invariant set in $V_L \cup V_R$ is hyperbolic.

6. Let f be a diffeomorphism and $\mathbf{p}$ a periodic point for f. Let $H_{\mathbf{p}}$ be the equivalence class of periodic points which are heteroclinically-related to $\mathbf{p}$. (These points are called

homclinically related by Newhouse.) Let $\Lambda_{\mathbf{p}} = \mathrm{cl}(H_{\mathbf{p}})$. Prove that f is topologically transitive on $\Lambda_{\mathbf{p}}$.

7 Assume that Λ is an attractor for f with a hyperbolic structure. Let $\mathbf{p} \in \Lambda$. Prove that $W^u(\mathbf{p}, f) \subset \Lambda$.