Geometry/Topology Exam

Student Id:____________________

Instructions: Complete each problem, show your work in detail. Theorems which are used or quoted must be stated explicitly.

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**Question 1.** Let $X$ be a connected three-dimensional topological manifold and $x, y \in X$ be two distinct points. Show that the inclusion $X \setminus y \subset X$ induces an isomorphism $\pi_1(X \setminus y, x) \rightarrow \pi_1(X, x)$.

**Question 2.**

a. Show that every continuous map $S^2 \rightarrow S^1$ is homotopy equivalent to the constant map.

b. Show that every principal circle bundle over $S^3$ is trivial.

**Question 3.** Let $(M^n, g)$ be a complete Riemannian manifold with $\gamma : [0, d] \rightarrow M^n$ a minimizing geodesic connecting $x$ and $y$:

a. Show for each $t < d$ that $\gamma : [0, t] \rightarrow M$ is a minimizing geodesic.

b. Show for each $t < d$ that $\gamma : [0, t] \rightarrow M$ is the unique minimizing geodesic from $x$ to $\gamma(t)$.

c. Give an example of $M$ and $\gamma$ showing $\gamma : [0, d] \rightarrow M$ may not be a unique minimizing geodesic.

**Question 4.** Let $(M^n, g)$ be a complete manifold with $p \in M$ and $d_p(x) = d(p, x)$ the distance function from $p$, so that $|\nabla d_p| = 1$.

a. For any $x \in M$ such that $d_p(x)$ is smooth near $x$, show that the hessian satisfies $\nabla^2 d_p(\nabla d_p, X) = 0$ for any $X \in T_x M$.

b. Show that if $A$ is $k \times k$ matrix then $|A|^2 \geq \frac{1}{k} (tr A)^2$, where $tr A$ is the trace of $A$ and $|A|^2 = \sum_{ij} A_{ij}^2$ is its norm squared.

c. Let $\gamma : [0, t] \rightarrow M$ be a minimizing geodesic from $p$ and define $m(t) \equiv \Delta d_p(\gamma(t))$. Show for $t \in (0, t)$ that $\frac{d}{dt} m(t) \leq -\frac{1}{n-1} m^2(t) - Rc(\dot{\gamma}, \dot{\gamma})$, where $\Delta = g^{ij} \nabla_i \nabla_j$ is the laplacian and $Rc$ is the Ricci curvature of the manifold.

d. Conclude that if $Rc \geq 0$ then whenever $d_p$ is smooth we have $\Delta d_p \leq \frac{n-1}{d_p}$. (Indeed, this holds in the barrier or distributional sense on all of $M$, but you don’t need to prove this)

**Question 5.** Let $M$ be a compact connected three-dimensional manifold without boundary.

a. Prove that the Euler characteristic $\chi(M)$ is zero, even if $M$ is not orientable (you may use the fact that $\chi(M) = \sum_{i=0}^3 (-1)^i dim(H_i(M, F))$ for any field $F$. That is, the right hand side is independent of $F$).

b. Assume $M$ is non-orientable, prove that $dim(H_1(M, \mathbb{Q})) > 0$ (Hint: Compute $\chi(M)$).

c. Assume $M$ is non-orientable, prove that $H_1(M, \mathbb{Z})$ is an infinite group.

**Question 6.**

a. Describe a CW complex whose underling space is $\mathbb{RP}^n$.

b. Following the definition of cell cohomology, compute $H^k(\mathbb{RP}^4, \mathbb{Z})$ for all $k$. 