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Probability Prelim—September 2006

Instructions: Do all of numbers 1-4. Choose exactly three from numbers 5-9.

1. State and prove the first and second Borel-Cantelli lemmas.

2. State and prove a suitable form of the strong law of large numbers for the sequence S_n/n, where $S_n := X_1 + \cdots + X_n$ and $X_n, n \geq 1$ is a sequence of independent random variables with $E|X_n|^4 \leq 1, \forall n \geq 1$.

3a. State and prove Kolmogorov's inequality for a sequence of independent random variables $X_n, n \geq 1$ with zero means and finite variances.
3b. Use this to show that $\sum_{n \geq 1} X_n$ converges a.s. if it converges in L^2.

4a. If $\phi(t)$ is the characteristic function of some random variable, prove that each of the following are characteristic functions: $\phi(-t)$, $\operatorname{Re}\phi(t)$, $\phi(t)\phi(-t)$, $\exp[\phi(t) - 1]$.
4b. State the continuity theorem for characteristic functions and use it to prove the central limit theorem for a sequence of i.i.d. random variables (X_n) with $EX_n = 0$, $\mathrm{Var}X_n = 1$.

5a. Define what is meant by the statement that the random variables $X_n, n \geq 1$ form a *martingale* with respect to the filtration $\{\mathcal{F}_n\}$.
5b. State and prove Kolmogorov's inequality for L^2 martingales and its application to the a.s convergence of L^2 martingales.

6a. Define what is meant by a *stopping time* relative to a filtration of sigma fields $\mathcal{F}_n, n \geq 1$.
6b. If $X_n, n \geq 1$ is a martingale and T a bounded stopping time relative to the filtration $\mathcal{F}(X_1, \ldots, X_n)$, prove that $E[X_T] = E[X_0]$.

7a. Define what is meant by the statement that the random variables $X_n, n \geq 0$ have the *Markov property.*
7b. Show that this is equivalent to the statement that, conditional on the present–the past and future are independent.

8a. Define what is meant by the statement that the random variables $X_n, n \geq 1$ form a *stationary sequence.*
8b. Show that this is equivalent, via the shift operator, to a measurable mapping which is measure-preserving.

9a. Define what is meant by *standard Brownian motion* $X_t, 0 \leq t \leq 1$.
9b. Prove that for any standard Brownian motion $X_t, 0 \leq t \leq 1$, we have the a.s. convergence

$$\lim_{N \to \infty} \sum_{k=1}^{2^N} [X_{kt2^{-N}} - X_{(k-1)t2^{-N}}]^2 = t, \qquad 0 \leq t \leq 1.$$