ICM 2014: The Structure and Meaning of Ricci Curvature

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Outline of Talk

- Background of Ricci Curvature and Limit Spaces
- Structure of Spaces with Lower Ricci Curvature
- Regularity of Spaces with Bounded Ricci Curvature
- Characterizing Ricci Curvature
Background: Curvatures

- \((M^n, g, x)\) n-dimensional pointed Riemannian Manifold.

Curvature: \(Rm(X, Y)Z \equiv \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z\)
- Should be interpreted as \textit{Hessian} of the metric

Ricci: \(Rc(X, Y) \equiv \sum \langle Rm(E_a, X) Y, E_a \rangle = tr(Rm)\)
- Should be interpreted as \textit{Laplacian} of the metric

Normalized Volume Measure: \(\nu \equiv \frac{dvg}{Vol(B_1(x))}\).
Background: Limit Spaces

- $(M^n_i, g_i, \nu_i, x_i) \overset{mGH}{\rightarrow} (X, d, \nu, x)$, convergence in measured Gromov-Hausdorff topology.

Theorem (Gromov 81')

Let $(M^n_i, g_i, x_i)$ be a sequence of Riemannian manifolds with $Rc_i \geq -\lambda g$. Then there exists a metric space $(X, d, x)$, such that after passing to a subsequence we have that

$$(M^n_i, g_i, x_i) \overset{GH}{\rightarrow} (X, d, x).$$

- Initiated study of Ricci limit spaces.
- Generalized by Fukaya and others to say that

$$(M^n_i, g_i, \nu_i, x_i) \overset{mGH}{\rightarrow} (X, d, \nu, x).$$

- Question: What is the structure of $X$?
Structure of Limit Spaces, Lower Ricci Curvature

Background:

Theorem (Cheeger-Colding 96')

Let \((M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)\) where \(Rc_i \geq -\lambda g\). Then for \(\nu\)-a.e. \(x \in X\) the tangent cone at \(x\) is unique and isometric to \(\mathbb{R}^{k_x}\) for some \(0 \leq k_x \leq n\).

Conjecture (Cheeger-Colding 96')

There exists \(0 \leq k \leq n\) such that \(k_x \equiv k\) is independent of \(x \in X\). In particular, \(X\) has a well defined dimension.

Conjecture (96')

The isometry group of \(X\) is a Lie Group.
Structure of Limit Spaces, Lower Ricci Curvature: Geometry of Geodesics

- Proof requires new understanding of the geometry of geodesics:

**Theorem (Colding-Naber 10')**

Let \((M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)\) where \(Rc_i \geq -\lambda g\), and let \(\gamma : [0, 1] \rightarrow X\) be a minimizing geodesic. Then there exists \(C(n, \delta, \lambda), \alpha(n) > 0\) such that for every \(s, t \in [\delta, 1 - \delta]\) and \(r \leq 1\) we have

\[
d_{GH}\left(B_r(\gamma(s)), B_r(\gamma(t))\right) \leq \frac{C}{\delta} |t - s|^\alpha r. \tag{2}
\]

- In words, the geometry of balls along geodesics can change at most at a Hölder rate.

- Corollary: By taking \(r \rightarrow 0\) we see that tangent cones change continuously along geodesics.
Structure of Limit Spaces, Lower Ricci Curvature: Applications

Theorem (Colding-Naber 10’)

Let \((M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)\) where \(Rc_i \geq -\lambda g\). Then there exists \(0 \leq k \leq n\) and a full measure subset \(R(X) \subseteq X\) such that the tangent cone at each point is unique and isometric to \(\mathbb{R}^k\).

- Can define \(dim(X) \equiv k\)
- Proof through the example of the trumpet space:

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- Pick a geodesic \(\gamma\) connecting the \(x\)-points.
Theorem (Colding-Naber 10’)

Let \((M^n_i, g_i, \nu_i, x_i) \xrightarrow{GH} (X, d, \nu, x)\) where \(Rc_i \geq -\lambda g\). Then there exists \(0 \leq k \leq n\) and a full measure subset \(R(X) \subseteq X\) such that the tangent cone at each point is unique and isometric to \(\mathbb{R}^k\).

- Can define \(\text{dim}(X) \equiv k\)
- Proof through the example of the trumpet space:

Tangent cones along \(\gamma(t)\) discontinuous at \(o\), hence the trumpet space cannot arise as a Ricci limit space.
Hölder continuity of tangent cones allow for further refinements about geometry of the regular set:

**Theorem (Colding-Naber 10')**

The regular set $R(X)$ is weakly convex. In particular, $R(X)$ is connected.

The convexity of $R(X)$ is the key point in showing the Isometry group conjecture:

**Theorem (Colding-Naber 10')**

*The isometry group of $X$ is a Lie Group.*

Proof by contradiction. Push *small subgroups* into a regular tangent cone, namely $\mathbb{R}^k$.

Produces small subgroups of the isometry group of $\mathbb{R}^k$, contradiction.
### Structure of Limit Spaces, Lower Ricci Curvature

**Open Questions:**

<table>
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<tr>
<th>Conjecture</th>
<th>Open Question</th>
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<tr>
<td><em>dim(X) is equal to the Hausdorff dimension of X.</em></td>
<td><em>Does the singular set $S(X) \equiv X \setminus R(X)$ have $dim(X) - 1$-Hausdorff dimension?</em></td>
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<td><em>Is there an open dense subset of $X$ which is homeomorphic to a manifold?</em></td>
<td><em>Is there an open dense subset of $X$ which is bilipschitz to a manifold?</em></td>
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Structure of Limit Spaces, Bounded Ricci Background:

**Theorem (Anderson,Bando,Kasue,Nakajima,Tian 89’)**

Let \((M^4_i, g_i) \rightarrow (X, d)\) satisfy \(\text{diam}(M_i) \leq D, \ |Rc_i| \leq 3, \ \text{Vol}(M^4_i) > v > 0\) and \(|b_2(M_i)| \leq A\), then \(X\) is a Riemannian orbifold with isolated singularities.

**Conjecture (Codimension Four Conjecture 89’)**

Let \((M^4_i, g_i, p_i) \rightarrow (X, d, p)\) satisfy \(|Rc_i| \leq n - 1\) and \(\text{Vol}(B_1(p_i)) > v > 0\), then \(X\) is a Riemannian manifold away from a set of codimension four.

**Conjecture (Anderson 94’)**

Let \((M^4, g)\) satisfy \(\text{diam}(M_i) \leq D, \ |Rc_i| \leq 3, \ \text{Vol}(M^4) > v > 0\), then \(M^4\) is one of \(C(D, v)\)-diffeomorphism types.
The key to solving the codimension four conjecture is to rule out codimension two singularities.

That is, if an Einstein manifold $M$ is close to $\mathbb{R}^{n-2} \times C(S^1_r)$:

- Then $M$ is actually smoothly close to Euclidean space $\mathbb{R}^n$.
- Key new idea is the slicing theorem:

**Theorem (Cheeger-Naber 14’)**

For each $\epsilon > 0$ there exists $\delta > 0$ such that if $u : B_2(x) \rightarrow \mathbb{R}^{n-2}$ is a harmonic $\delta$-splitting, then there exists $G_\epsilon \subseteq B_1(0^{n-2})$ with $|B_1 \setminus G_\epsilon| < \epsilon$ such that: if $s \in G_\epsilon$ and $x \in u^{-1}(s) \cap B_1(x)$ with $0 < r < 1$, then $\exists A_r \in GL(n-2)$ such that $Au : B_r(x) \rightarrow \mathbb{R}^{n-2}$ is an $\epsilon$-splitting.
Structure of Limit Spaces
Bounded Ricci, codimension two singularities:

- Proof of the slicing theorem is by far the most involved aspect of the proof.
- The level sets $u^{-1}(s)$ are approximations of the cone factors.
- Slicing theorem allows one to pick *good* cone slices and reblow up to arrive at a new Ricci limit space which is a smooth cone:

\[ \mathbb{R}^{n-2} \times \text{cone} \]

- From this picture it is easy to contradict the existence of the upper Ricci bound, and thus the existence of codimension two singularities.
An easier argument works for codim 3 singularities. Combining yields:

**Theorem (Cheeger-Naber 14')**

Let \((M^n, g, p)\) satisfy \(\text{Vol}(B_1(p)) > v > 0\). Then there exists \(\epsilon(n, v) > 0\) such that if \(|Rc| \leq \epsilon\) and if

\[
d_{GH}(B_2(p), B_2(y)) < \epsilon,
\]

where \(y \in \mathbb{R}^{n-3} \times C(Y)\), then the harmonic radius satisfies \(r_h(x) \geq 1\). If \(M^n\) is Einstein then we further have

\[
\sup_{B_1(p)} |Rm| \leq 1. \tag{3}
\]
By combining the previous $\epsilon$-regularity theorem with the stratification theory one obtains:

**Corollary (Cheeger-Naber 14’)**

Let $(M_i^4, g_i, p_i) \xrightarrow{GH} (X, d, p)$ satisfy $|Rc_i| \leq n - 1$ and $Vol(B_1(p_i)) > \nu > 0$, then there exists $S(X) \subseteq X$ with $\dim S(X) \leq n - 4$ such that $R(X) \equiv X \setminus S(X)$ is a Riemannian manifold.
More generally, by combining the previous $\epsilon$-regularity theorem with the quantitative stratification theory one obtains the effective estimates:

**Theorem (Cheeger-Naber 14’)**

Let $(M^n, g, p)$ satisfy $|Rc| \leq n - 1$ and $\text{Vol}(B_1(p)) > \nu > 0$. Then for every $\epsilon > 0$ there exists $C_\epsilon(n, \nu, \epsilon)$ such that

$$\int_{B_1(p)} |Rm|^{2-\epsilon} \leq C_\epsilon. \quad (4)$$

Furthermore, for every $r \leq 1$ we have the harmonic radius estimate

$$\text{Vol}(B_r \{ x : r_h(x) \leq r \} \cap B_1(p)) \leq C_\epsilon r^{4-\epsilon}. \quad (5)$$
These results may be pushed further in dimension four.
As a starting point consider the following:

**Theorem (Cheeger-Naber 14')**

If \((M^4_i, g_i, p_i) \xrightarrow{GH} (X, d, p)\) where \(|Rc_i| \leq 3\) and \(Vol(B_1(p_i)) > v > 0\), then \(X\) is a Riemannian orbifold with isolated singularities.

This in turn may be used to prove Anderson’s conjecture:

**Theorem (Cheeger-Naber 14')**

If \((M^4, g)\) such that \(|Rc| \leq 3\), \(Vol(B_1(p)) > v > 0\), and \(diam(M) \leq D\). Then there exists \(C(v, D)\) such that \(M\) has at most one of \(C\)-diffeomorphism types.
A local version of the finite diffeomorphism type theorem, when combined with the Chern-Gauss-Bonnet, gives rise to the following:

**Theorem (Cheeger-Naber 14')**

\[
\text{Let } (M^4, g, p) \text{ satisfy } |Rc| \leq 3 \text{ and } \text{Vol}(B_1(p)) > v > 0. \text{ Then there exists } C(v) \text{ such that}
\]

\[
\int_{B_1(p)} |Rm|^2 \leq C. \quad (6)
\]

- This estimate is sharp
Structure of Limit Spaces, Bounded Ricci Open Questions:

**Conjecture**

Let \((M^n, g, p)\) satisfy \(|\text{Rc}| \leq n - 1\) and \(\text{Vol}(B_1(p)) > v > 0\). Then there exists \(C(n, v)\) such that \(\int_{B_1(p)} |Rm|^2 \leq C\).

**Corollary**

Let \((M^n_i, g_i, p_i) \stackrel{GH}{\longrightarrow} (X, d, p)\) with \(|\text{Rc}_i| \leq n - 1\) and \(\text{Vol}(B_1(p_i)) > v > 0\), then the singular set \(S(X)\) is \(n-4\)-rectifiable with \(H^{n-4}(S(X) \cap B_1(p)) \leq C(n, v)\).

**Conjecture**

Let \((M^n_i, g_i, p_i) \stackrel{GH}{\longrightarrow} (X, d, p)\) with \(|\text{Rc}_i| \leq n - 1\) and \(\text{Vol}(B_1(p_i)) > v > 0\), then \(X\) is bilipschitz to a real analytic variety.
Future directions in Ricci curvature will involve more than a regularity theory.

Many ways to interpret the *meaning* of Ricci curvature bounds.

Each new method leads to new understanding.

At this stage there are many methods for characterizing lower Ricci curvature (see next slide).

Want to characterize and understand the meaning of *bounded* Ricci curvature.
Characterizing Ricci Curvature, Background:  
Lower Ricci Curvature

Theorem (Bakry-Emery-Ledoux 85’)

Let \((M^n, g)\) be a complete manifold, then the following are equivalent:

1. \(\text{Ric} \geq -\kappa g\).
2. \(|\nabla H_t u|(x) \leq e^{\frac{k}{2}t} H_t |\nabla u|(x) \ \forall x\).
3. \(\lambda_1(-\Delta_{x,t}) \geq \kappa^{-1} \left( e^{\kappa t} - 1 \right)\)
4. \(\int_M u^2 \ln u^2 \rho_t(x, dy) \leq 2\kappa^{-1} \left( e^{\kappa t} - 1 \right) \int_M |\nabla u|^2 \rho_t(x, dy)\) if \(\int_M u^2 \rho_t = 1\).

- \(H_t\) heat flow operator, \(\rho_t\) heat kernel,
- \(\Delta_{x,t} = \Delta + \nabla \ln \rho_t \cdot \nabla\) heat kernel laplacian.
- Lower Bounds on Ricci \(\iff\) Analysis on \(M\)
- More recently: lower ricci \(\iff\) convexity of the entropy functional (Lott, Villani, Sturm, Ambrosio, Gigli, Saviere’).
Characterizing Ricci Curvature: Bounded Ricci Curvature

- Characterizations of bounded Ricci curvature will require estimates on the path space $P(M)$ of the manifold.
- There will be a 1-1 correspondence between the B-E-L estimates and the new estimates on path space.
- In fact, for each estimate on path space, we will see how when it is applied to the simplest functions on path space we recover the BEL estimates. Namely, $F(\gamma) = u(\gamma(t))$.
- We will see that Bounded Ricci Curvature $\iff$ Analysis on Path Space of $M$. 

Characterizing Bounded Ricci Curvature: Path Space Basics

- $P(M) \equiv C^0([0, \infty), M)$
- $P_x(M) \equiv \{ \gamma \in P(M) : \gamma(0) = x \}$.

For a partition $t \equiv \{0 \leq t_1 < \ldots < t_k < \infty \}$ denote by $e_t : P(M) \to M^k$ the evaluation mapping given by

$$e_t(\gamma) = (\gamma(t_1), \ldots, \gamma(t_k)).$$

For $x \in M$ let $\Gamma_x$ be the associated Wiener measure on $P(M)$. Defined by its pushforwards:

$$e_{t,*} \Gamma_x = \rho_{t_1}(x, dy_1) \rho_{t_2-t_2}(y_1, dy_2) \cdots \rho_{t_k-t_{k-1}}(y_{k-1}, dy_k).$$
If $F : P(M) \to \mathbb{R}$ we define the Parallel Gradient:

$$|\nabla_0 F|_{\gamma} = \sup\{DVF : |V|(0) = 1 \text{ and } |\nabla_{\dot{\gamma}} F| \equiv 0\}.$$

If $F : P(M) \to \mathbb{R}$ we define the $t$-Parallel Gradient:

$$|\nabla_t F|_{\gamma} = \sup\{DVF : |V|(s) = 0 \text{ for } s < t, |V|(t) = 1 \text{ and } |\nabla_{\dot{\gamma}} F|(s) \equiv 0 \text{ for } s > t\}.$$
Characterizing Bounded Ricci Curvature: First Characterization, Gradient Bounds:

- Given $F : P(M) \rightarrow \mathbb{R}$ let us construct a function on $M$ by
  $$\int_{P(M)} Fd\Gamma_x : M \rightarrow \mathbb{R}.$$  

- If $F \in C(P(M))$ then $\int Fd\Gamma_x \in C(M)$.

- What about gradient bounds? Do gradient bounds on $F$ give rise to gradient bounds on $\int Fd\Gamma_x$? In fact:
  $$|\nabla \int_{P(M)} Fd\Gamma_x| \leq \int_{P(M)} |\nabla_0 F|d\Gamma_x$$  

  ⇔  

  $$Rc \equiv 0$$
Let us apply this to the simplest functions on path space. For $t > 0$ fixed and $u : M \to \mathbb{R}$ let $F(\gamma) = u(\gamma(t))$.

Let us compute $|\nabla \int_{P(M)} Fd\Gamma_x| \leq \int_{P(M)} |\nabla_0 F|d\Gamma_x$:

1. $\int_{P(M)} Fd\Gamma_x \equiv H_t u(x)$.
2. $|\nabla_0 F|_{(\gamma)} = |\nabla u|_{(\gamma(t))}$.
3. Thus

$$|\nabla \int_{P(M)} Fd\Gamma_x| \leq \int_{P(M)} |\nabla_0 F|d\Gamma_x \quad \forall F \in L^2(P(M))$$

$\downarrow$

$$|\nabla H_t u|(x) \leq H_t|\nabla u|(x) \quad \forall u \in L^2(M)$$

- Recover Bakry-Emery, hence Ric $\geq 0$. 

Recall $L^2(P(M))$ comes naturally equipped with a one-parameter family of closed nested subspaces $L_t^2 \subseteq L^2(P(M))$.

- $F \in L_t^2$ if $F(\gamma) = F(\sigma)$ whenever $\gamma|_{[0,t]} = \sigma|_{[0,t]}$.

- Given $F$ can construct a family of functions $F_t \in L_t^2 \subseteq L^2(P(M))$ by projection.

- $F_t$ is a martingale. As a curve in $L^2$, $F_t$ is precisely $C^{1/2}$. 
Characterizing Bounded Ricci Curvature: Second Characterization:

To understand $C^{1/2}$-derivative define Quadratic Variation

$$[F_t] \equiv \lim_{t \in [0,t]} \sum \frac{(F_{t_{a+1}} - F_{t_a})^2}{t_{a+1} - t_a}$$

Can we control the derivative of $[F_t]$? In fact:

$$\frac{d}{dt} [F_t](\gamma) \leq \int_{P_{\gamma(t)}(M)} |\nabla_t F|$$

$\Updownarrow$

$Rc \equiv 0$.

Similar statements for $|Rc| \leq k$, metric measure spaces, and dimensional versions.
Recall that the Ornstein-Uhlenbeck operator
\[ L_x : L^2(P_x(M)) \rightarrow L^2(P_x(M)) \]
is a self adjoint operator on based path space.

Arises from the Dirichlet Form
\[ E[F] \equiv \int_{P_x(M)} |\nabla_{H^1} F|^2 d\Gamma_x = \int_{P_x(M)} \int_0^\infty |\nabla_s F|^2 d\Gamma_x, \]
where \( \nabla_{H^1} F \) is the Malliavin gradient.

Acts as an infinite dimensional laplacian. Spectral gap first proved by Gross in \( \mathbb{R}^n \), and Aida and K. D. Elworthy for general compact manifolds. Fang and Hsu first proved estimates using Ricci curvature.
More generally one can define the time restricted Dirichlet energies \( E_{t_0}^{t_1} [F] \equiv \int_{P_x(M)} \int_{t_0}^{t_1} |\nabla_s F|^2 d\Gamma_x \).

Thus \( E_{0}^{\infty} \equiv E \), and in general \( E_{t_0}^{t_1} \) is the part of the Dirichlet energy which only sees the gradient on the time range \([t_0, t_1]\).

From these energies one can define the induced Ornstein-Uhlenbeck operators \( L_{t_0}^{t_1} : L^2(P_x(M)) \to L^2(P_x(M)) \) with \( L_{0}^{\infty} \equiv L_x \).
Characterizing Bounded Ricci Curvature: Third Characterization:

- Is the spectrum of the operators $L_{t_0}^{t_1}$ controlled or characterized by Ricci Curvature? In fact:

$$\int_{P(M)} |F_{t_1} - F_{t_0}|^2 \leq \int_{P(M)} \langle F, L_{t_0}^{t_1} F \rangle$$

$\updownarrow$

$$Rc \equiv 0.$$

In particular, we have the spectral gap $\lambda(L_x) \geq 1$ for the standard Ornstein-Uhlenbeck operator.

- More generally there are log-Sobolev versions of this result, as well as similar statements for $|Rc| \leq k$, metric measure spaces, and dimensional versions.
Characterizing Bounded Ricci Curvature:

Below is a partial list of the main results, see [N] for the complete statement:

**Theorem (Naber 13')**

Let \( (M^n, g) \) be a smooth Riemannian manifold, then the following are equivalent:

1. \(-\kappa g \leq \text{Ric} \leq \kappa g.\)
2. \[|\nabla \int_{P(M)} F \, d\Gamma_x| \leq \int_{P(M)} \left( |\nabla_0 F| + \int_0^\infty \frac{\kappa}{2} e^{\kappa t} |\nabla_t F| \, dt \right) \, d\Gamma_x.\]
3. \[\int_{P(M)} |F_{t_1} - F_{t_0}|^2 \leq e^{\frac{\kappa}{2}(T-t_0)} \int_{P(M)} \langle F, L_{t_0,\kappa}^t F \rangle, \text{ in particular}\]
   \[\lambda^1(L^T_x) \geq \frac{2}{e^{\kappa T+1}} \text{ for the standard Ornstein-Uhlenbeck operator.}\]
4. \[\frac{d}{dt} [F_t](\gamma) \leq e^{\kappa(T-t)} \int_{P_{\gamma(t)}(M)} |\nabla_t F| + \int_t^T \frac{\kappa}{2} e^{\frac{\kappa}{2}s} |\nabla_s F|^2 \, d\Gamma_{\gamma(t)}\]

where \( F \) is an \( \cal{F}^T \)-measurable function.