1. Problem Session 1

Exercise 1. *Show that for $x > 0$*

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{x^2}{2}}\, dx \leq \frac{1}{x}\, e^{-\frac{x^2}{2}}.$$

Exercise 2.

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{\sqrt{2\pi}} \int_{a\sqrt{n}}^{b\sqrt{n}} e^{-\frac{t^2}{2}}\, dt\right) = -\inf_{x\in[a,b]} \frac{x^2}{2}.$$

Exercise 3. *Find a candidate for the LDP rate function for μ_n if $(X_i)_{i\in\mathbb{N}}$ are i.i.d.*

(a) *exponential random variables with parameter $\lambda > 0$.*
(b) *Poisson random variables with parameter $\lambda > 0$.*
(c) *Is there a relationship between the two rate functions (associated to the same λ)?*

Hint: *If $(N_t)_{t\geq 0}$ is a Poisson process with rate λ and τ_n is the time of the n-th arrival then $N_t = \sup\{n \geq 0 :\ \tau_n \leq t\}$.*

Answer:

$$\mathcal{I}_{Exp}(x) = \begin{cases} \lambda x - \ln(\lambda x) - 1, & \text{if } x > 0; \\ \infty, & \text{if } x \leq 0; \end{cases}$$

$$\mathcal{I}_{Poi}(x) = \begin{cases} x\ln(x/\lambda) - (x - \lambda), & \text{if } x > 0; \\ \lambda, & \text{if } x = 0; \\ \infty, & \text{if } x < 0, \end{cases}$$

Exercise 4. *Which is more likely: getting no heads when a fair coin is tossed n times, or getting fewer than n heads when a fair coin is tossed $8n$ times?*

Exercise 5. *Which is more likely: getting no heads when a fair coin is tossed n times, or getting fewer than n heads when a fair coin is tossed $4n$ times?*

Exercise 6. *Assume that $X_1, X_2, \ldots, X_n$ are independent random variables such that $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$ where $p_i \in (0,1)$ for every $i \in \{1, 2, \ldots, n\}$. Let $x > \mathbb{E}[X_1 + X_2 + \cdots + X_i]$. Prove that*

$$\mathbb{P}(X_1 + X_2 + \cdots + X_n > x) < \frac{\left((p_1 + p_2 + \cdots + p_n) \cdot \frac{e}{x}\right)^x}{e^{p_1 + \cdots + p_n}}.$$

Exercise 7.

(a) *If $x > 0$ and $y \in (0,1)$ are real numbers, prove that*

$$e^{\frac{x^2}{2} + xy} \geq ye^x + 1 - y.$$

(b) *Assume that $X_1, X_2, \ldots, X_n$ are independent random variables such that $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$ where $p_i \in (0,1)$ for every $i \in \{1, 2, \ldots, n\}$. Let $\mu = \mathbb{E}[X_1 + X_2 + \cdots + X_i]$. Prove that for each $\varepsilon > 0$ the following holds*

$$\mathbb{P}(X_1 + X_2 + \cdots + X_n > \mu + \varepsilon) \leq e^{-\frac{\varepsilon^2}{2n}}.$$