Problem Session 3

The following two exercises produce parts of a proof of Varadhan-Bryc theorem.

Exercise 1. *Prove that if a sequence of probability measures $(\mu)_{n\in\mathbb{N}}$ on $(\mathbb{X},\mathcal{X})$ satisfies a LD lower bound with a rate function $\mathcal{I}$ and F is a lower-semicontinuous function on $\mathbb{X}$, then*

$$\lim_{n\to\infty}\frac{1}{n}\ln\int_{\mathbb{X}} e^{nF}\,d\mu_n \geq \sup_{x\in\mathbb{X}}\left(F(x)-\mathcal{I}(x)\right).$$

Hint: for each $x\in\mathbb{X}$ and $\epsilon>0$ consider an open set $O_{x,\epsilon} := \{y\in\mathbb{X} : F(y) > F(x)-\epsilon\}$, restrict the integral to this set to set a lower bound. Then let $\epsilon\to 0$.

Exercise 2. *Given a rate function $\mathcal{I}$ and a sequence of probability measures $(\mu)_{n\in\mathbb{N}}$ on $(\mathbb{X},\mathcal{X})$, show that if for every lower-semicontinuous function F*

$$\lim_{n\to\infty}\frac{1}{n}\ln\int_{\mathbb{X}} e^{nF}\,d\mu_n \geq \sup_{x\in\mathbb{X}}\left(F(x)-\mathcal{I}(x)\right). \tag{1}$$

then $(\mu_n)_{n\in\mathbb{N}}$ satisfies a LD lower bound with a rate function $\mathcal{I}$.
Hint: given an open set O, for $x\in O$, a small $\delta>0$, and large $N>0$ define $F_{x,\delta,N}(y) = -N\min\left\{\frac{d(x,y)}{\delta},1\right\}$. For $F_{x,\delta,N}$ estimate the left hand side of (1) above and below, and then let $N\to\infty$. Notation: $d(x,y)$ is the distance between x and y in $\mathbb{X}$.

Y is said to have a *lognormal* distribution with parameters $\mu\in\mathbb{R}$ and $\sigma>0$ if $\ln Y$ is normal with mean μ and variance σ^2.

Exercise 3. *Let $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of lognormal random variables with parameters μ and σ and ν_n be the distribution of geometric means, $\tilde{Y}_n = \left(\prod_{i=1}^n Y_i\right)^{1/n}$. Is there a LDP for ν_n? If yes, then what is the rate function?*

Exercise 4. *Use the contraction principle to derive Cramér theorem on a finite probability space from Sanov theorem (see the second lecture).*

Exercise 5. *Let $(\mu_n)_{n\in\mathbb{N}}$ be the distribution of empirical means of a sequence of i.i.d. Bernoulli variables with parameter $1/2$. Use exponential tilting to obtain a LDP for the distributions $(\nu_n)_{n\in\mathbb{N}}$ of empirical means of a sequence of i.i.d. Bernoulli variables with parameter $p\neq 1/2$ directly from the LDP for $(\mu_n)_{n\in\mathbb{N}}$ (see the first lecture).*

In preparation for the last lecture, solve the following exercise.

Exercise 6. *Let μ be a probability measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ with a density proportional to $e^{-|x|}/(1+|x|^{d+2})$ and Λ be its logarithmic MGF.*
(a) Show that $D_\Lambda = \{x\in\mathbb{R}^d : \|x\|\leq 1\}$.
(b) Show that $|\nabla\Lambda(x)| \not\to \infty$ as $x\in D_\Lambda$ approaches the boundary of D_Λ.
This is an example of non-steep logarithmic MGF.

Exercise 7. *Study the Curie-Weiss model discussed in lecture notes in the case when $h\neq 0$.*