TENTATIVE PROBLEM SET FOR THE COURSE ON THE STOCHASTIC BURGERS EQUATION.

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Proofs of some simple facts useful for understanding the lectures are moved to the problem sessions. There will be two problem sessions. The problems are split into sections not by sessions but by topics. We'll see how it goes. Please report misprints to us.

1. Simpler random dynamical systems. One Force – One Solution Principle

1. Let $(W_n)_{n\in\mathbb{Z}}$ be i.i.d. r.v.'s. Let $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a Borel map. Prove that if X_0 is independent of $\sigma(W_1,W_2,\ldots)$, then the process $(X_n)_{n\geq 0}$ defined inductively by $X_n=F(X_{n-1},W_n)$, $n\geq 1$ is a Markov process. [Notes, p.98]
2. Consider a Markov process on $\mathbb{R}$ with 1-step transition probability function
$$P(x,A)=\int_A \frac{1}{\sqrt{2\pi}\sigma}e^{-(y-ax)^2/(2\sigma^2)}dy,\quad x\in\mathbb{R},\ A\in\mathcal{B}(\mathbb{R}),$$
where $a\in(0,1)$, $\sigma>0$, prove that the distribution $\nu=\mathcal{N}(0,\frac{\sigma^2}{1-a^2})$ is invariant, the Markov process with initial condition distributed as ν is stationary, and for any initial distribution μ, $\mu P^n\Rightarrow\nu$.
3. Consider a random dynamical system on $\mathbb{R}^1$ given by
$$\phi_{n,\omega}(x)=ax+\sigma W_n(\omega),\quad n\in\mathbb{Z},\ x\in\mathbb{R},$$
where $a\in(0,1)$, $\sigma>0$ and $(W_n)_{n\in\mathbb{Z}}$ are i.i.d. standard Gaussian r.v.'s. Prove that the stochastic process $(X_n)_{n\in\mathbb{Z}}$ given by
$$X_n=\sigma\sum_{k=0}^{\infty}a^kW_{n-k}$$
is the only stationary global solution, i.e., it is a unique (up to zero-measure modifications) stationary process satisfying
$$X_n=\phi_{n,\omega}(X_{n-1}),\quad n\in\mathbb{Z}.$$
See [Notes, p.97]
4. For vectors $x,y\in\mathbb{R}^n$, with nonnegative entries, the Hilbert projective metric is defined by
$$\Theta(x,y)=-\ln\left(\min_i\frac{x_i}{y_i}\cdot\min_j\frac{y_j}{x_j}\right)=\ln\left(\max_i\frac{y_i}{x_i}\cdot\max_j\frac{x_j}{y_j}\right)=\ln\max_{i,j}\frac{x_jy_i}{y_jx_i}.$$
Let Δ be the simplex of all vectors $x\in\mathbb{R}^n$ with nonnegative entries and such that $|x|_1=x_1+\ldots+x_n=1$.
   Suppose a matrix A has all positive entries. We can define $Tx=Ax/|Ax|_1$ and denote
$$D=\operatorname{diam}(T\Delta)=\sup\{\Theta(x,y):x,y\in\Delta\}<\infty,$$
Prove that for all $x,y\in\Delta$,
$$\Theta(Tx,Ty)\leq\frac{1-e^{-D/2}}{1+e^{-D/2}}\Theta(x,y).$$
See [Notes, p.72]

2. Burgers equation

5. Suppose $\phi=\phi(t,x)$ solves
$$\partial_t\phi=\frac{\varkappa}{2}\partial_{xx}\phi-\frac{F\phi}{\varkappa},$$
Then the Hopf–Cole transformation $U=-\varkappa\ln\phi$ gives a solution of the HJ equation
$$\partial_tU+\frac{(\partial_xU)^2}{2}=\frac{\varkappa}{2}\partial_{xx}U+F,$$

and $u = \partial_x U$ solves the Burgers equation

$$\partial_t u + u\partial_x u = \frac{\varkappa}{2}\partial_{xx} u + \partial_x F.$$

6. Assuming smoothness of the solution of the Burgers equation

$$\partial_t u + u\partial_x u = \frac{\varkappa}{2}\partial_{xx} u + \partial_x F$$

on the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, prove that

$$\int_{\mathbb{T}^1} u(t,x)dx = const.$$

7. The Hopf–Cole formula solving the unforced Burgers equation can be written as

$$u_\varkappa(t,x) = \int_{\mathbb{R}} \frac{x-y}{t}\mu_{t,x,\varkappa}(dy),$$

where

$$\mu_{t,x,\varkappa}(dy) = \frac{e^{-\frac{(x-y)^2}{2\varkappa} - \frac{U(0,y)}{\varkappa}}dy}{Z_{t,x,\varkappa}},$$

$Z_{t,x,\varkappa}$ being the normalization factor making $\mu_{t,x,\varkappa}$ a probability measure. Prove that $\mu_{t,x,\varkappa}$ is stochastically dominated by $\mu_{t,x',\varkappa}$ if $x < x'$. Derive that

$$x - tu_\varkappa(t,x)$$

is monotone in x. (Set $t = \varkappa = 1$ if that makes your life easier.)

3. Action minimizers and polymers

8. For a path $\gamma : \{m, m+1, \ldots, n\} \to \mathbb{R}$ its action is defined by

$$A^{m,n}(\gamma) = \frac{1}{2}\sum_{k=m}^{n-1}(\gamma_{k+1} - \gamma_k)^2 + \sum_{k=m}^{n-1} F_k(\gamma_k).$$

A path γ is called a minimizer on $[m,n]$ if for any path γ' with $\gamma'_m = \gamma_m$ and $\gamma'_n = \gamma_n$, one has $A^{m,n}(\gamma') \geq A^{m,n}(\gamma)$. Prove that if a sequence of minimizers (γ^i) converges to a path γ pointwise as $i \to \infty$, then γ is also a minimizer.

9. Prove that every action minimizer satisfies the following Euler–Lagrange equation: $\gamma_{k+1} - \gamma_k = \gamma_k - \gamma_{k-1} + \partial_x F_k(x)$

10. For any $m, n \in \mathbb{Z}$ with $m < n$ and any $x, y \in \mathbb{R}$, the point-to-point polymer measure $\mu^{m,n}_{x,y}$ is introduced via its density

$$p^{m,n}_{x,y}(x_m, \ldots, x_n) = \frac{\prod_{k=m}^{n-1}\left[g_{2\varkappa}(x_{k+1} - x_k)e^{-\frac{F_k(x_k)}{2\varkappa}}\right]}{Z^{m,n}_{x,y}},$$

with respect to $\delta_x \times \mathrm{Leb}^{n-m-1} \times \delta_y$, where Leb is the Lebesgue measure on $\mathbb{R}$,

$$g_D(x) = \frac{1}{\sqrt{2\pi D}}e^{-\frac{x^2}{2D}}, \quad x \in \mathbb{R},$$

and $Z^{m,n}_{x,y}$ is a normalizing constant called the partition function.

A measure μ on the set of paths $S^{m,n}_{x,*}$ is called a (point-to-measure) polymer measure if there is a probability measure ν on $\mathbb{R}$ such that $\mu = \mu^{m,n}_{x,\nu}$, where

$$\mu^{m,n}_{x,\nu} = \int_{\mathbb{R}} \mu^{m,n}_{x,y}\nu(dy).$$

Prove that the projection of $\mu^{m,n}_{x,\nu}$ on $\{m, m+1, \ldots, k\}$, where $m < k \leq n$ is also a point-to-measure polymer measure.

11. Prove that if a sequence of point-to-measure polymer measures has a weak limit, the limit is also a polymer measure.